FFTX SPIRAL Backend

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ECP EXASCALE COMPUTING PROJECT

Have You Ever Wondered About This?

Numerical Linear Algebra

LAPACK
ScaLAPACK
LU factorization
Eigensolves
SVD

BLAS, BLACS
BLAS-1
BLAS-2
BLAS-3

Spectral Algorithms

Convolution
Correlation
Upsampling
Poisson solver
…

?

FFTW
DFT, RDFT
1D, 2D, 3D,…
batch

No LAPACK equivalent for spectral methods

- **Medium size 1D FFT (1k—10k data points) is most common library call**
  applications break down 3D problems themselves and then call the 1D FFT library
- **Higher level FFT calls rarely used**
  FFTW *guru* interface is powerful but hard to used, leading to performance loss
- **Low arithmetic intensity and variation of FFT use make library approach hard**
  Algorithm specific decompositions and FFT calls intertwined with non-FFT code

FFTX and SpectralPACK

Numerical Linear Algebra		Spectral Algorithms
LAPACK LU factorization Eigensolves SVD …	≈	**SpectralPACK** Convolution Correlation Upsampling Poisson solver …
BLAS BLAS-1 BLAS-2 BLAS-3		**FFTX** DFT, RDFT 1D, 2D, 3D,… batch

Define the LAPACK equivalent for spectral algorithms

- **Define FFTX as the BLAS equivalent**
  provide user FFT functionality as well as algorithm building blocks
- **Define class of numerical algorithms to be supported by SpectralPACK**
  PDE solver classes (Green's function, sparse in normal/k space,…), signal processing,…
- **Library front-end, code generation and vendor library back-end**
  mirror concepts from FFTX layer

FFTX and SpectralPACK solve the "spectral motif" long term

Example: Poisson's Equation in Free Space

Partial differential equation (PDE)

$$\Delta(\Phi) = \rho$$

$$\rho : \mathbb{R}^3 \to \mathbb{R}$$

$$D = \mathrm{supp}(\rho) \subset \mathbb{R}^3$$

Poisson's equation. Δ is the Laplace operator

Solution characterization

$$\Phi : \mathbb{R}^3 \to \mathbb{R}$$

$$\Phi(\vec{x}) = \frac{Q}{4\pi||\vec{x}||} + o\left(\frac{1}{||\vec{x}||}\right) \text{ as } ||\vec{x}|| \to \infty$$

$$Q = \int_D \rho d\vec{x}$$

Approach: Green's function

$$\Phi(\vec{x}) = \int_D G(\vec{x} - \vec{y})\rho(\vec{y})d\vec{y} \equiv (G * \rho)(\vec{x}), \quad G(\vec{x}) = \frac{1}{4\pi||\vec{x}||_2}$$

Solution: $\phi(.)$ = convolution of RHS $\rho(.)$ with Green's function $G(.)$. Efficient through FFTs (frequency domain)

Method of Local Corrections (MLC)

$$\tilde{G}_k = \frac{1}{4\pi||k - N\vec{u}||_2^2} \quad \text{if } k \neq N\vec{u}$$

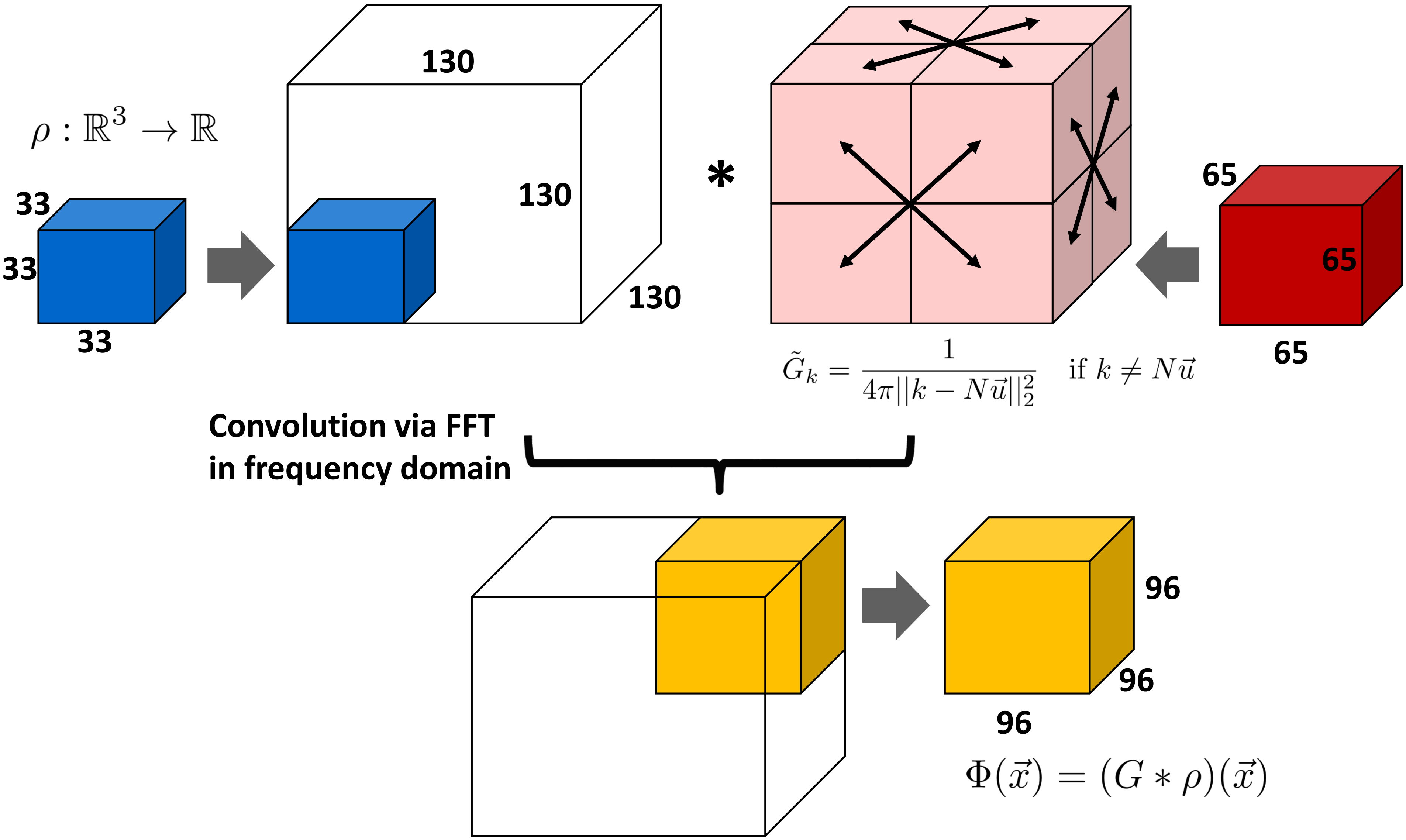
Green's function kernel in frequency domain

P. McCorquodale, P. Colella, G. T. Balls, and S. B. Baden: **A Local Corrections Algorithm for Solving Poisson's Equation in Three Dimensions.** Communications in Applied Mathematics and Computational Science Vol. 2, No. 1 (2007), pp. 57-81., 2007.

C. R. Anderson: **A method of local corrections for computing the velocity field due to a distribution of vortex blobs.** Journal of Computational Physics, vol. 62, no. 1, pp. 111–123, 1986.

Algorithm: Hockney Free Space Convolution

$\rho : \mathbb{R}^3 \to \mathbb{R}$



$$\tilde{G}_k = \frac{1}{4\pi||k - N\vec{u}||_2^2} \quad \text{if } k \neq N\vec{u}$$

$$\Phi(\vec{x}) = (G * \rho)(\vec{x})$$

Hockney: Convolution + problem specific zero padding and output subset

FFTX C++ Code: Hockney Free Space Convolution

```
box_t<3> inputBox(point_t<3>({{0,0,0}}),point_t<3>({32,32,32}));
array_t<3, double> rho(inputBox);
// ... set input values.

box_t<3> transformBox(point_t<3>({{0,0,0}}),point_t<3>({{129,129,129}}));
box_t<3> outputBox(point_t<3>({33,33,33}),point_t<3>({129,129,129}));

point_t<3> kindF({{DFT,DFT,DFT}});

size_t normalize = normalization(transformBox);

auto forward_plan =
    plan_dft<3,double,std::complex<double>(kindF,inputBox,transformBox,transformBox);

auto kernel_plan = kernel<3,std::complex<double> >(greensFunction, transformBox, normalize);

point_t<3> kindI({{IDFT,IDFT,IDFT}});
auto inverse_plan = plan_dft<3, std::complex<double >, double>
                       (kindI, transformBox, outputBox, transformBox);

auto solver = chain(chain(forward_plan,kernel_plan),inverse_plan);

context_t context;
context_omp(context, 8);

std::ofstream splFile("hockney.spl");
export_spl(context, solver, splFile, "hockney33_97_130");
splFile.close();
// Offline codegen.
auto fptr = import_spl<3, double, double>("hockney33_97_130");
array_t<3, double> Phi(inputBox);
fptr(&rho, &Phi, 1);
```

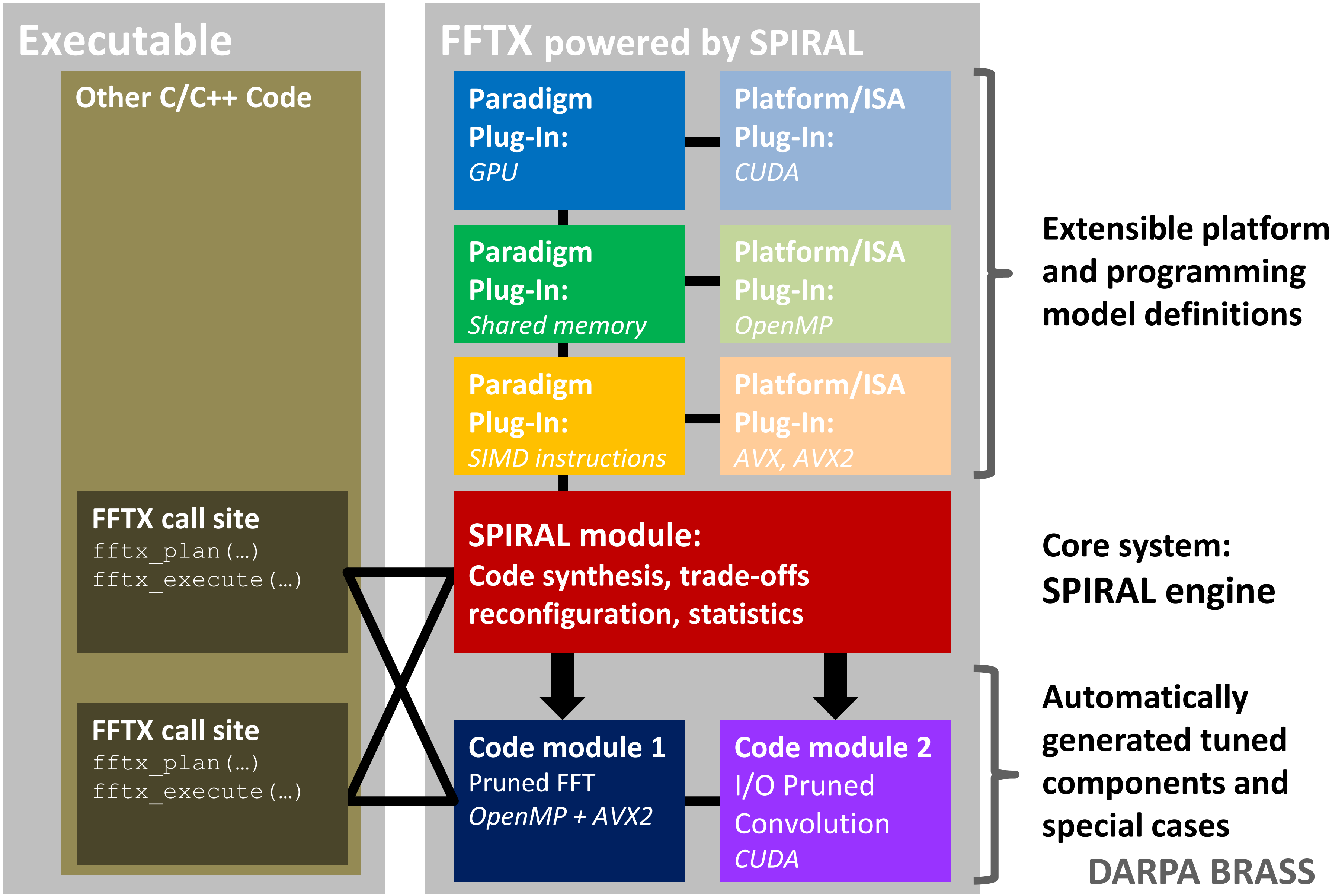
FFTX C++ Code: Hockney Free Space Convolution

```
box_t<3> inputBox(point_t<3>({{0,0,0}}),point_t<3>({32,32,32}));
array_t<3, double> rho(inputBox);
// ... set input values.

box_t<3> transformBox(point_t<3>({{0,0,0}}),point_t<3>({{129,129,129}})));
box_t<3> outputBox(point_t<3>({33,33,33}),point_t<3>({129,129,129}));

point_t<3> kindF({{DFT,DFT,DFT}});

size_t

auto fo
    pla

auto ke                                                                   lize);

point_t
auto in


auto so


context
context

std::ofstream splFile("hockney.spl");
export_spl(context, solver, splFile, "hockney33_97_130");
splFile.close();
// Offline codegen.
auto fptr = import_spl<3, double, double>("hockney33_97_130");
array_t<3, double> Phi(inputBox);
fptr(&rho, &Phi, 1);
```

This is a specification dressed as a program

- Needs to be clean and concise
- No code level optimizations and tricks
- Don't think "performance" but "correctness"
- *For production code and software engineering*

FFTX Backend: SPIRAL



SPIRAL 8.2.0: Available Under Open Source

- **Open Source SPIRAL available**
  - non-viral license (BSD)
  - Initial version, effort ongoing to open source whole system
  - Commercial support via SpiralGen, Inc.
- **Developed over 20 years**
  - Funding: DARPA (OPAL, DESA, HACMS, PERFECT, BRASS), NSF, ONR, DoD HPC, JPL, DOE, CMU SEI, Intel, Nvidia, Mercury
- **Open sourced under DARPA PERFECT, continuing under DOE ECP**
- **Tutorial material available online**

www.spiral.net

F. Franchetti, T. M. Low, D. T. Popovici, R. M. Veras, D. G. Spampinato, J. R. Johnson, M. Püschel, J. C. Hoe, J. M. F. Moura: SPIRAL: Extreme Performance Portability, **Proceedings of the IEEE, Vol. 106, No. 11, 2018.**
Special Issue on *From High Level Specification to High Performance Code*

FFTX Program Trace

```
Load(fftx);
ImportAll(fftx);

conf := FFTXGlobals.confHockneyMlcCUDADevice();
opts := FFTXGlobals.getOpts(conf);

t := let(
    n := 130, ns := 33, nd := 96,
    name := "hockney"::StringInt(n)::"_"::StringInt(nd)::"_"::StringInt(ns),
    symvar := var("symbl", TPtr(TReal)),
    TFCall(
        Compose([
            ExtractBox([n,n,n], [[n-nd..n-1],[n-nd..n-1],[n-nd..n-1]]),
            IMDPRDFT([n,n,n], 1),
            RCDiag(FDataOfs(symvar, 2*n*n*(n/2+1), 0)),
            MDPRDFT([n,n,n], -1),
            ZeroEmbedBox([n,n,n], [[0..ns-1],[0..ns-1],[0..ns-1]])]),
        rec(fname := name, params := [symvar])
    ).withTags(opts.tags)
);

c := opts.fftxGen(t);
opts.prettyPrint(c);
```

FFTX Program Trace

```
Load(fftx);
ImportAll(fftx);

conf := FFTXGlobals.confHockneyMlcCUDADevice();
op

t
        fcc(fname := name, params := [symvar])
    ).withTags(opts.tags)
);

c := opts.fftxGen(t);
opts.prettyPrint(c);
```

The whole convolution kernel is captured

- DAG with all dependencies
- User-defined call-backs
- Captures prunining, zero-padding and symmetries
- *Lifts sequence of C/C++ library calls to a specification*

SPIRAL Options Capture Performance Engineering

```
hockneyMlcCUDADeviceOpts := function(arg)
    local opts;
    opts := Copy(HockneyMlcCUDADeviceOpts);
    opts.includes := [];
    opts.breakdownRules.PRDFT := [ PRDFT1_Base2, CopyFields(PRDFT_PD, rec(maxSize := 7)), PRDFT_PD_loop];
    opts.breakdownRules.IPRDFT := [ IPRDFT1_Base1, IPRDFT1_Base2, IPRDFT_PD_loop, IPRDFT_PD ];
    opts.breakdownRules.IPRDFT2 := List([ IPRDFT2_Base1, IPRDFT2_Base2, IPRDFT2_CT], _noT);
    opts.breakdownRules.PRDFT3 := [ ];
    opts.breakdownRules.URDFT := List([ URDFT1_Base1, URDFT1_Base2, URDFT1_Base4, URDFT1_CT ], _noT);
    opts.breakdownRules.DFT := List([ DFT_Base, CopyFields(DFT_PD, rec(maxSize := 7)),  DFT_PD_loop], _noT);
    opts.breakdownRules.PrunedPRDFT := [ PrunedPRDFT_base, PrunedPRDFT_CT_rec_block ];
    opts.breakdownRules.PrunedIPRDFT := [ CopyFields(PrunedIPRDFT_base, PrunedIPRDFT_CT_rec_block ];
    opts.breakdownRules.PrunedDFT := List([ PrunedDFT_base, PrunedDFT_CT_rec_block ], _noT);
    opts.breakdownRules.IOPrunedMDRConv := List([ IOPrunedMDRConv_3D_2trip_zyx_freqdata ], _noT);
    opts.breakdownRules.MDRConv := [ MDRConv_3D_2trip_zyx_freqdata ];

    opts.formulaStrategies.postProcess := opts.formulaStrategies.postProcess
        :: opts.formulaStrategies.sigmaSpl :: [MergedRuleSet(HockneyMLC_SIMTRules, RulesSums)];


    opts.preProcess := (self, t) >> let(t1 := RulesFFTXPromoteNT(Copy(t)), RulesFFTXPromoteNT_Cleanup(t1));

    return opts;
end;
```

SPIRAL Options Capture Performance Engineering

```
hockneyMlcCUDADeviceOpts := function(arg)
    local opts;
    opts := Copy(HockneyMlcCUDADeviceOpts);
    opts.includes := [];
    opts.breakdownRules.PRDFT := [ PRDFT1_Base2, CopyFields(PRDFT_PD, rec(maxSize := 7)), PRDFT_PD_loop];
    opts.breakdownRules.IPRDFT := [ IPRDFT1_Base1, IPRDFT1_Base2, IPRDFT_PD_loop, IPRDFT_PD ];
    opts.breakdownRules.IPRDFT2 := List([ IPRDFT2_Base1, IPRDFT2_Base2, IPRDFT2_CT], _noT);
```

Configures code generation

- Developed by performance engineer + application specialist
- Lifts FFTX call sequence into SPIRAL non-terminal
- Configures all SPIRAL rewriting systems
- Does code generation and autotuning
- *Clear separation of concerns frontend/backend*

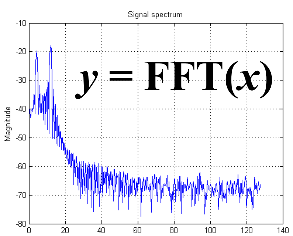
SPIRAL: Go from Mathematics to Software

Given:

- **Mathematical problem specification**
  core mathematics does not change
- **Target computer platform**
  varies greatly, new platforms introduced often

Wanted:

- **Very good implementation of specification on platform**
- **Proof of correctness**



on

automatic

```
void fft64(double  *Y, double  *X) {
    ...
    s5674 = _mm256_permute2f128_pd(s5672, s5673, (0) | ((2) << 4));
    s5675 = _mm256_permute2f128_pd(s5672, s5673, (1) | ((3) << 4));
    s5676 = _mm256_unpacklo_pd(s5674, s5675);
    s5677 = _mm256_unpackhi_pd(s5674, s5675);
    s5678 = *((a3738 + 16));
    s5679 = *((a3738 + 17));
    s5680 = _mm256_permute2f128_pd(s5678, s5679, (0) | ((2) << 4));
    s5681 = _mm256_permute2f128_pd(s5678, s5679, (1) | ((3) << 4));
    s5682 = _mm256_unpacklo_pd(s5680, s5681);
    s5683 = _mm256_unpackhi_pd(s5680, s5681);
    t5735 = _mm256_add_pd(s5676, s5682);
    t5736 = _mm256_add_pd(s5677, s5683);
    t5737 = _mm256_add_pd(s5670, t5735);
    t5738 = _mm256_add_pd(s5671, t5736);
    t5739 = _mm256_sub_pd(s5670, _mm256_mul_pd(_mm_vbroadcast_sd(&(C22)), t5735));
    t5740 = _mm256_sub_pd(s5671, _mm256_mul_pd(_mm_vbroadcast_sd(&(C22)), t5736));
    t5741 = _mm256_mul_pd(_mm_vbroadcast_sd(&(C23)), _mm256_sub_pd(s5677, s5683));
    t5742 = _mm256_mul_pd(_mm_vbroadcast_sd(&(C23)), _mm256_sub_pd(s5676, s5682));
    ...
}
```

performance

QED.

Inspiration: Symbolic Integration

- **Rule based AI system**
  basic functions, substitution
- **May not succeed**
  not all expressions can be symbolically integrated
- **Arbitrarily extensible**
  define new functions as integrals
  Γ(.), distributions, Lebesgue integral
- **Semantics preserving**
  rule chain = formal proof
- **Automation**
  Mathematica, Maple

Table of Integrals

BASIC FORMS

(1) $\int x^n dx = \frac{1}{n+1} x^{n+1}$

(2) $\int \frac{1}{x} dx = \ln x$

(3) $\int u dv = uv - \int v du$

(4) $\int u(x) v'(x) dx = u(x)v(x) - \int v(x) u'(x) dx$

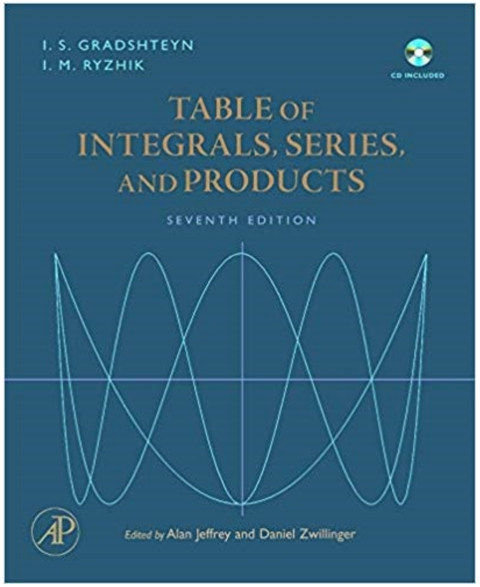
RATIONAL FUNCTIONS

(5) $\int \frac{1}{ax+b} dx = \frac{1}{a} \ln(ax+b)$

(6) $\int \frac{1}{(x+a)^2} dx = \frac{-1}{x+a}$

(7) $\int (x+a)^n dx = (x+a)^n \left(\frac{a}{1+n} + \frac{x}{1+n} \right), \; n \neq -1$

(8) $\int x(x+a)^n dx = \frac{(x+a)^{1+n}(nx+x-a)}{(n+2)(n+1)}$



In[31]:= $\int_0^{2\pi} \frac{1}{a^2 \mathrm{Cos}[t]^2 + b^2 \mathrm{Sin}[t]^2} \, dt$

Out[31]= $\frac{2\sqrt{\frac{b^2}{a^2}}\,\pi}{b^2}$

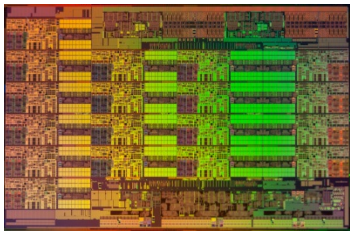
Wolfram Mathematica

In[33]:= $\int_0^{2\pi} \frac{1}{a^2 \left(\frac{e^{i t} + e^{-i t}}{2}\right)^2 + b^2 \left(\frac{e^{i t} - e^{-i t}}{2 i}\right)^2} \, dt$

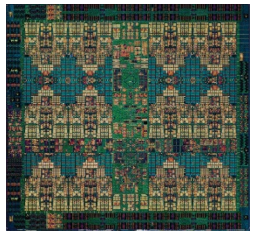
Out[33]= 0

SPIRAL’s Target Computing Landscape

1 Gflop/s = one billion floating-point operations (additions or multiplications) per second

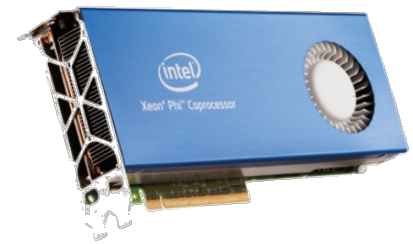


Intel Xeon 8180M
2.25 Tflop/s, 205 W
28 cores, 2.5—3.8 GHz
2-way—16-way AVX-512



IBM POWER9
768 Gflop/s, 300 W
24 cores, 4 GHz
4-way VSX-3

Nvidia Tesla V100
7.8 Tflop/s, 300 W
5120 cores, 1.2 GHz
32-way SIMT



Intel Xeon Phi 7290F
1.7 Tflop/s, 260 W
72 cores, 1.5 GHz
8-way/16-way LRBni

Snapdragon 835
15 Gflop/s, 2 W
8 cores, 2.3 GHz
A540 GPU, 682 DSP, NEON

Intel Atom C3858
32 Gflop/s, 25 W
16 cores, 2.0 GHz
2-way/4-way SSSE3

Dell PowerEdge R940
3.2 Tflop/s, 6 TB, 850 W
4x 24 cores, 2.1 GHz
4-way/8-way AVX

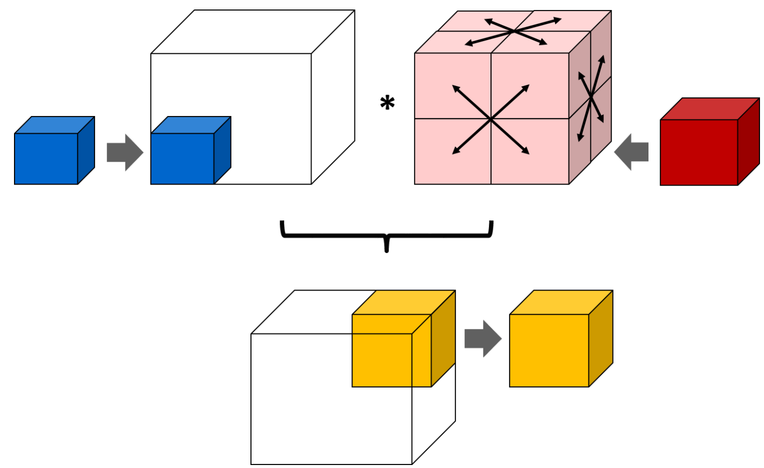
Summit
187.7 Pflop/s, 13 MW
9,216 x 22 cores POWER9
+ 27,648 V100 GPUs

Rules in Internal Domain Specific Language

Linear Transforms

$$
\begin{aligned}
\mathbf{DFT}_n &\to (\mathbf{DFT}_k \otimes \mathrm{I}_m)\, \mathrm{T}^n_m (\mathrm{I}_k \otimes \mathbf{DFT}_m)\, \mathrm{L}^n_k, \quad n = km\\
\mathbf{DFT}_n &\to P_n(\mathbf{DFT}_k \otimes \mathbf{DFT}_m) Q_n, \quad n = km,\ \gcd(k,m)=1\\
\mathbf{DFT}_p &\to R_p^T(\mathrm{I}_1 \oplus \mathbf{DFT}_{p-1}) D_p (\mathrm{I}_1 \oplus \mathbf{DFT}_{p-1}) R_p, \quad p \text{ prime}\\
\mathbf{DCT\text{-}3}_n &\to (\mathrm{I}_m \oplus \mathrm{J}_m)\, \mathrm{L}^n_m (\mathbf{DCT\text{-}3}_m(1/4) \oplus \mathbf{DCT\text{-}3}_m(3/4))\\
&\quad \cdot (\mathrm{F}_2 \otimes \mathrm{I}_m) \begin{bmatrix} \mathrm{I}_m & 0 \oplus -\mathrm{J}_{m-1} \\ & \frac{1}{\sqrt{2}}(\mathrm{I}_1 \oplus 2\,\mathrm{I}_m) \end{bmatrix}, \quad n = 2m\\
\mathbf{DCT\text{-}4}_n &\to S_n \mathbf{DCT\text{-}2}_n\, \mathrm{diag}_{0 \le k < n}(1/(2\cos((2k+1)\pi/4n)))\\
\mathbf{IMDCT}_{2m} &\to (\mathrm{J}_m \oplus \mathrm{I}_m \oplus \mathrm{I}_m \oplus \mathrm{J}_m)\left(\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \otimes \mathrm{I}_m\right) \oplus \left(\begin{bmatrix} -1 \\ -1 \end{bmatrix} \otimes \mathrm{I}_m\right)\right) \mathrm{J}_{2m}\, \mathbf{DCT\text{-}4}_{2m}\\
\mathbf{WHT}_{2^k} &\to \prod_{i=1}^{t} (\mathrm{I}_{2^{k_1+\cdots+k_{i-1}}} \otimes \mathbf{WHT}_{2^{k_i}} \otimes \mathrm{I}_{2^{k_{i+1}+\cdots+k_t}}), \quad k = k_1 + \cdots + k_t\\
\mathbf{DFT}_2 &\to \mathrm{F}_2\\
\mathbf{DCT\text{-}2}_2 &\to \mathrm{diag}(1, 1/\sqrt{2})\, \mathrm{F}_2\\
\mathbf{DCT\text{-}4}_2 &\to \mathrm{J}_2\, \mathrm{R}_{13\pi/8}
\end{aligned}
$$

Spectral Domain Algorithms



Hardware

- Multithreading (Multicore) — $\mathrm{I}_p \otimes_{\parallel} A_{\mu n}, \quad \mathrm{L}^{mn}_m \bar{\otimes} \mathrm{I}_\mu$
- Vector SIMD (SSE, VMX/Altivec,...) — $A \hat{\otimes} \mathrm{I}_\nu, \quad \underbrace{\mathrm{L}^{2\nu}_2}_{\text{isa}}, \quad \underbrace{\mathrm{L}^{2\nu}_\nu}_{\text{isa}}, \quad \underbrace{\mathrm{L}^{\nu^2}_\nu}_{\text{isa}}$
- Message Passing (Clusters, MPP) — $\mathrm{I}_p \otimes_{\parallel} A_n, \quad \underbrace{\mathrm{L}^{p^2}_p \bar{\otimes} \mathrm{I}_{n/p^2}}_{\text{all-to-all}}$
- Streaming/multibuffering (Cell) — $\mathrm{I}_n \otimes_2 A_{\mu n}, \quad \mathrm{L}^{mn}_m \bar{\otimes} \mathrm{I}_\mu$
- Graphics Processors (GPUs) — $\prod_{i=0}^{n-1} A_i, \quad A_n \bar{\otimes} \mathrm{I}_w, \quad P_n \otimes Q_w$
- Gate-level parallelism (FPGA) — $\prod_{i=0}^{n-1}{}^{\mathrm{ir}} A, \quad \mathrm{I}_s \tilde{\otimes} A, \quad \underbrace{\mathrm{L}^m_n}_{\text{bram}}$
- HW/SW partitioning (CPU + FPGA) — $\underbrace{A_1}_{\text{fpga}}, \quad \underbrace{A_2}_{\text{fpga}}, \quad \underbrace{A_3}_{\text{fpga}}, \quad \underbrace{A_4}_{\text{fpga}}$

Program Transformations

$$
\begin{aligned}
\underbrace{AB}_{\mathrm{smp}(p,\mu)} &\to \underbrace{A}_{\mathrm{smp}(p,\mu)}\ \underbrace{B}_{\mathrm{smp}(p,\mu)}\\
\underbrace{A_m \otimes \mathrm{I}_n}_{\mathrm{smp}(p,\mu)} &\to \underbrace{\left(\mathrm{L}^{mp}_m \otimes \mathrm{I}_{n/p}\right)\left(\mathrm{I}_p \otimes (A_m \otimes \mathrm{I}_{n/p})\right)\left(\mathrm{L}^{mp}_p \otimes \mathrm{I}_{n/p}\right)}_{\mathrm{smp}(p,\mu)}\\
\underbrace{\mathrm{L}^{mn}_m}_{\mathrm{smp}(p,\mu)} &\to \begin{cases} \underbrace{\left(\mathrm{I}_p \otimes \mathrm{L}^{mn/p}_{m/p}\right)}_{\mathrm{smp}(p,\mu)} \underbrace{\left(\mathrm{L}^{pn}_p \otimes \mathrm{I}_{m/p}\right)}_{\mathrm{smp}(p,\mu)} \\ \underbrace{\left(\mathrm{L}^{pm}_m \otimes \mathrm{I}_{n/p}\right)}_{\mathrm{smp}(p,\mu)} \underbrace{\left(\mathrm{I}_p \otimes \mathrm{L}^{mn/p}_m\right)}_{\mathrm{smp}(p,\mu)} \end{cases}
\end{aligned}
$$

Recursive rules

$$
\begin{aligned}
\underbrace{\mathrm{I}_m \otimes A_n}_{\mathrm{smp}(p,\mu)} &\to \mathrm{I}_p \otimes_{\parallel} \left(\mathrm{I}_{m/p} \otimes A_n\right)\\
\underbrace{(P \otimes \mathrm{I}_n)}_{\mathrm{smp}(p,\mu)} &\to \left(P \otimes \mathrm{I}_{n/\mu}\right) \bar{\otimes} \mathrm{I}_\mu
\end{aligned}
$$

Base case rules

Autotuning in Constraint Solution Space

$\underbrace{\mathrm{DFT}_8}_{\mathrm{AVX}(2\text{-way } \mathbb{C})}$

AVX 2-way
`_Complex double`

DFT_8

Base cases

$A^{n\times n}\vec{\otimes} \mathrm{I}_2$

$\underbrace{\mathrm{L}_2^4}_{\mathrm{vec}(2)}$

$\underbrace{\mathrm{T}_n^{mn}}_{\mathrm{vec}(2)}$

Transformation rules

$(\mathrm{I}_m \otimes A^{n\times n})\mathrm{L}_m^{mn} \to \left(\mathrm{I}_{m/\nu} \otimes \mathrm{L}_\nu^{n\nu}(A^{n\times n}\otimes \mathrm{I}_\nu)\right)(\mathrm{L}_{m/\nu}^{mn/\nu}\otimes \mathrm{I}_\nu)$

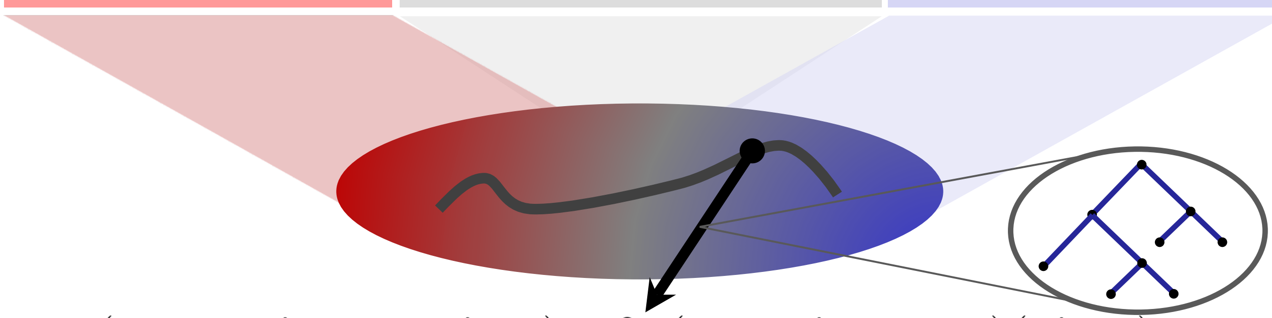
$\mathrm{L}_\nu^{n\nu} \to (\mathrm{L}_\nu^n \otimes \mathrm{I}_\nu)(\mathrm{I}_{n/\nu}\otimes \mathrm{L}_\nu^{\nu^2})$

$A^{m\times m}\otimes \mathrm{I}_n \to (A^{m\times m}\otimes \mathrm{I}_{n/\nu})\otimes \mathrm{I}_\nu$

Breakdown rules

$\mathrm{DFT}_{mn} \to (\mathrm{DFT}_m\otimes \mathrm{I}_n)\mathrm{T}_n^{mn}(\mathrm{I}_m\otimes \mathrm{DFT}_n)\mathrm{L}_m^{mn}$

$\mathrm{DFT}_2 \to \mathrm{F}_2$



$$\left((\mathrm{F}_2\otimes \mathrm{I}_2)\mathrm{T}_2^4(\mathrm{I}_2\otimes \mathrm{F}_2)\mathrm{L}_2^4\vec{\otimes} \mathrm{I}_2\right)\underbrace{\mathrm{T}_2^8}_{\mathrm{vec}(2)}\left(\mathrm{I}_2\otimes \underbrace{\mathrm{L}_2^4}_{\mathrm{vec}(2)}(\mathrm{F}_2\vec{\otimes} \mathrm{I}_2)\right)\left(\mathrm{L}_2^4\vec{\otimes} \mathrm{I}_2\right)$$

FFTX C/C++ specification code

↓ Inspector/executor

OL specification

↓ Expansion + backtracking

OL (dataflow) expression

↓ Recursive descent

Σ-OL (loop) expression

↓ Confluent term rewriting

Optimized Σ-OL expression

↓ Recursive descent

Abstract code

↓ Confluent term rewriting

Optimized abstract code

↓ Recursive descent

C code module

Nonterminal Expression (tSPL)

```
conf := FFTXGlobals.confHockneyMlcCUDADevice();
opts := FFTXGlobals.getOpts(conf);

t := let(
    n := 130, ns := 33, nd := 96,
    name := "hockney"::StringInt(n)::"_"::StringInt(nd)::"_"::StringInt(ns),
    symvar := var("symbl", TPtr(TReal)),
    TFCall(
        Compose([
            ExtractBox([n,n,n], [[n-nd..n-1],[n-nd..n-1],[n-nd..n-1]]),
            IMDPRDFT([n,n,n], 1),
            RCDiag(FDataOfs(symvar, 2*n*n*(n/2+1), 0)),
            MDPRDFT([n,n,n], -1),
            ZeroEmbedBox([n,n,n], [[0..ns-1],[0..ns-1],[0..ns-1]])]),
        rec(fname := name, params := [symvar])
    ).withTags(opts.tags)
);

## API definitions
_tofAdd := (n, l) -> When(IsList(l), let(mn := Minimum(l), mx := Maximum(l),
Checked(ForAll([mn..mx], i->i in l), fAdd(n, mx-mn+1, mn))),
                          When(n <> 1, fAdd(n, 1, 0), fId(n)));
_toBox := (ns, nps)-> When(IsList(ns), fTensor(List(Zip2(ns, nps), i->ApplyFunc(_tofAdd,
i))), _tofAdd(ns, nps));

ZeroEmbedBox := (ns, nps) -> Scat(_toBox(ns, nps));
ExtractBox := (ns, nps) -> Gath(_toBox(ns, nps));
```

DAG Nonterminal and Fortran Array Layout

```
Xcmj := tcast(TPtr(TColMaj(BoxND(szns, TComplex))), X);
Ycmj := tcast(TPtr(TColMaj(BoxND(szns, TComplex))), Y);

t := let(ns := szns,
    k := -1,
    name := "dft"::StringInt(Length(ns))::"d_cmj",
    TFCall(
        TDAG([
            TDAGNode(TRC(MDDFT(ns, k)), Ycmj, Xcmj)
        ]),
        rec(fname := name, params := [])).withTags(opts.tags)
);
```

Pruned Convolution Promotion Rules

```
Hockney_promote := ARule(Compose,
    [[@(6,Gath), @(8,fTensor, e->ForAll(e.children(), i->ObjId(i)=fAdd))],
     @(1,IMDPRDFT, e -> e.params[2] = 1), [@(2,RCDiag), @(4, FDataOfs, e->e.ofs = 0), @(5,I)],
     @(3,MDPRDFT, e -> e.params[2] = Product(e.params[1])-1),
     [@(7,Scat), @(9,fTensor, e->ForAll(e.children(), i->ObjId(i)=fAdd))]],
    e-> let(ii := Ind(Rows(@(2).val)), sym := @(2).val.element.var,
            symf := Lambda(ii, nth(sym,ii)),
            opat := List(@(8).val.children(), i-> _toSymList(List(i.tolist(), _unwrap))),
            ipat := List(@(9).val.children(), i-> _toSymList(List(i.tolist(), _unwrap))),
        [ IOPrunedMDRConv(@(1).val.params[1], symf, 1, opat, 1, ipat, true) ]))

Scat_Circulant_Gath__IOPrunedRConv := ARule(Compose,
    [[@(1, Gath), fAdd],  @(2,Circulant), [@(3, Scat), fAdd]],
    e-> [ IOPrunedRConv(@(2).val.params[1],
            FDataOfs(@(2).val.params[2].var, 2*(@(2).val.params[1]/2+1), 0),
            1, _toSymList(List(@(1).val.func.tolist(), _unwrap)),
            1, _toSymList(List(@(3).val.func.tolist(), _unwrap)), true)]),

MDDFT_Scat__PrunedMDDFT := ARule(Compose,
    [@(1, MDDFT), [@(2, Scat), @(3,fTensor, e->ForAll(e.children(), i->ObjId(i)=fAdd))]],
    e -> [ PrunedMDDFT(@(1).val.params[1], @(1).val.params[2], 1,
        List(@(3).val.children(), i-> _toSymList(List(i.tolist(), _unwrap))))] ),

Gath_MDDFT__PrunedIMDDFT := ARule(Compose,
    [[@(1, Gath), @(2,fTensor, e->ForAll(e.children(), i->ObjId(i)=fAdd))], @(3, MDDFT)],
    e -> [PrunedIMDDFT(@(3).val.params[1], @(3).val.params[2], 1,
        List(@(2).val.children(), i-> _toSymList(List(i.tolist(), _unwrap)))]),
```

Promoted Nonterminal

```
...
spiral> t;
TFCall(Gath(fTensor(fAdd(130, 96, 34), fAdd(130, 96, 34), fAdd(130, 96, 34))) *
  IMDPRDFT([ 130, 130, 130 ], 1) *
  RCDiag(FDataOfs(symbl, 2230800, V(0)), I(2)) *
  MDPRDFT([ 130, 130, 130 ], 2196999) *
  Scat(fTensor(fAdd(130, 33, 0), fAdd(130, 33, 0), fAdd(130, 33, 0))), rec(
  fname := "hockney130_96_33",
  params := [ symbl ] ))

spiral> tt := opts.preProcess(t);
TFCall(IOPrunedMDRConv([ 130, 130, 130 ], Lambda([ i8 ], nth(symbl, i8)), 1, [ [ 34 .. 129 ], [
34 .. 129 ], [ 34 .. 129 ] ], 1, [ [ 0 .. 32 ], [ 0 .. 32 ], [ 0 .. 32 ] ], true), rec(
  fname := "hockney130_96_33",
  params := [ symbl ] ))
```

Pruned Convolution Breakdown Rule

```
NewRulesFor(MDRConv, rec(
    MDRConv_3D_2trip_zyx_freqdata := rec(
        applicable :=  (self, nt) >> Length(nt.params[1]) = 3 and nt.params[3],
        children := nt -> let(....
            [[ PRDFT(nlist[1], -1),   # stage 1: PRDFT z
               DFT(nlist[2], -1),     # stage 2: DFT y
               CConv(nlist[3], hfunc), # stage 3+4+5: complex conv in x
               DFT(nlist[2], 1), # stage 6: iDFT in y
               IPRDFT(nlist[1], 1),    # stage 7: iPRDFT in z
            ]]),

        apply := (nt, C, cnt) -> let(...
            stage1 := L(2*nfreq*n3*n2, n3) * Tensor(I(n2), Tensor(L(2*nfreq, 2)
                          * prdft1d, I(n3))) * Tensor(L(n2*n1, n2), I(n3)),
            stage2 := Tensor(I(n3), Tensor(RC(pdft1d), I(nfreq))),
            pp := Tensor(L(n3*n2*nfreq, n2*nfreq), I(2)) * Tensor(I(n3), L(2*nfreq*n2, nfreq)),
            ppi := Tensor(I(n3), L(2*nfreq*n2, 2*n2)) * Tensor(L(n3*n2*nfreq, n3), I(2)),
            stage543 := Grp(ppi * IDirSum(i, RC(iopconv)) * pp),
            stage76 := Tensor(L(n2*n1, n1), I(n3)) * Grp(Tensor((Tensor(I(n2), iprdft1d
                 * L(2*nfreq, nfreq)) * Tensor(RC(ipdft1d), I(nfreq))), I(n3))
                 * L(2*nfreq*n3*n2, 2*nfreq*n2)),
            conv3dr := stage76 * stage543 * stage2 * stage1, conv3dr))
));
```

Expanded Nonterminal: RuleTree

```
spiral> rt := opts.search(tt);
TFCall_tag( TFCall(IOPrunedMDRConv([ 130, 130, 130 ], Lambda([ i8 ], nth(symbl, i8)), 1, [ Set([
34..129 ]), Set([ 34..129 ]), Set([ 34..129 ]) ], 1, [ Set([ 0..32 ]), Set([ 0.. 32 ]), Set([ 0..32 ])
], true), rec(fname := "hockney130_96_33", params := [ symbl ] )),
  IOPrunedMDRConv_3D_2trip_zyx_freqdata( IOPrunedMDRConv([ 130, 130, 130 ], Lambda([ i8 ], nth(symbl,
i8)), 1, [Set([ 34..129 ]), Set([ 34..129 ]), Set([ 34..129 ]) ], 1, [ Set([ 0..32 ]), Set([ 0.. 32 ]),
Set([ 0..32 ]) ], true),
    PrunedPRDFT_CT_rec_block( PrunedPRDFT(130, 129, 1, Set([ 0..32 ])),
      PRDFT1_Base2( PRDFT(2, 1) ),
      DFT_Base( DFT(2, 1) ),
      PrunedPRDFT_CT_rec_block( PrunedPRDFT(65, 64, 1, Set([ 0..16 ])),
        PRDFT_PD_loop( PRDFT(13, 12) ),
        DFT_PD_loop( DFT(13, 12) ),
        PrunedPRDFT_base( PrunedPRDFT(5, 4, 1, [ 0, 1 ]),
          PRDFT_PD( PRDFT(5, 4) ) ),
        PrunedPRDFT_base( PrunedPRDFT(5, 4, 1, [ 0 ]),
          PRDFT_PD( PRDFT(5, 4) ) ) ),
      PrunedPRDFT_CT_rec_block( PrunedPRDFT(65, 64, 1, Set([ 0..15 ])),
        PRDFT_PD_loop( PRDFT(13, 12) ),
        DFT_PD_loop( DFT(13, 12) ),
        PrunedPRDFT_base( PrunedPRDFT(5, 4, 1, [ 0, 1 ]),
          PRDFT_PD( PRDFT(5, 4) ) ),
        PrunedPRDFT_base( PrunedPRDFT(5, 4, 1, [ 0 ]),
          PRDFT_PD( PRDFT(5, 4) ) ) ) ),
    PrunedDFT_CT_rec_block( PrunedDFT(130, 129, 1, Set([ 0..32 ])),
      DFT_PD( DFT(5, 4) ),
      PrunedDFT_CT_rec_block( PrunedDFT(26, 25, 1, Set([ 0..6 ])),
        DFT_Base( DFT(2, 1) ),
        PrunedDFT_base( PrunedDFT(13, 12, 1, [ 0, 1, 2, 3 ]),
          DFT_PD_loop( DFT(13, 12) ) ),
        PrunedDFT_base( PrunedDFT(13, 12, 1, [ 0, 1, 2 ]),
          DFT_PD_loop( DFT(13, 12) ) ) ),
      PrunedDFT_CT_rec_block( PrunedDFT(26, 25, 1, Set([ 0, 1, 2, 3, 4, 5 ])),
        DFT_Base( DFT(2, 1) ),
        PrunedDFT_base( PrunedDFT(13, 12, 1, [ 0, 1, 2 ]),
          DFT_PD_loop( DFT(13, 12) ) ) ),
...
```

Translating an OL Expression Into Code

Constraint Solver Input: $\underbrace{\mathrm{DFT}_8}_{\mathrm{AVX(2\text{-}way}\ \mathbb{C})}$

Output =

Ruletree, expanded into

OL Expression:

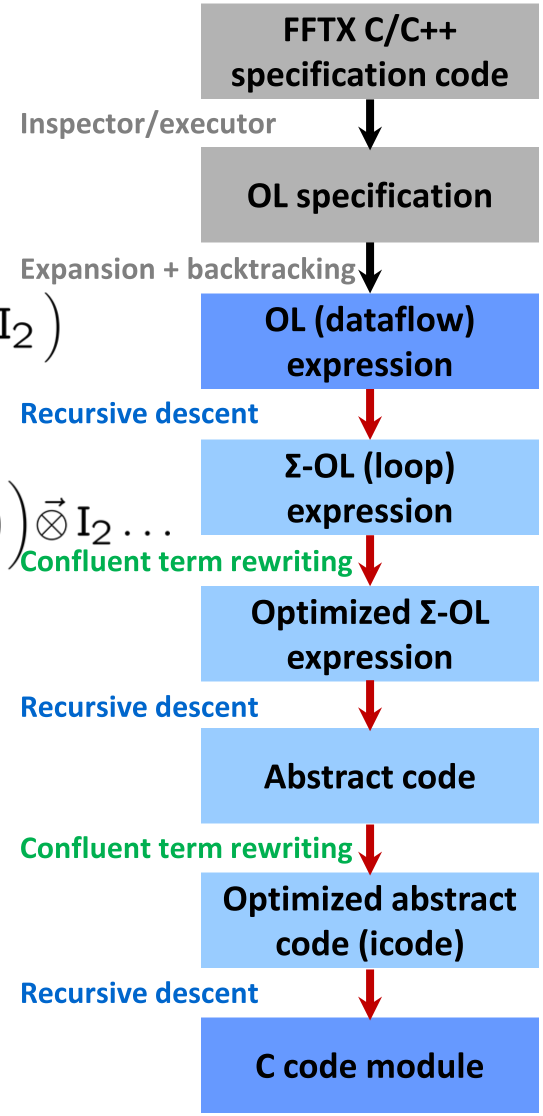
$$\left((F_2 \otimes I_2) T^4_2 (I_2 \otimes F_2) L^4_2 \vec{\otimes} I_2\right) \underbrace{T^8_2}_{\mathrm{vec}(2)} \left(I_2 \otimes \underbrace{L^4_2}_{\mathrm{vec}(2)} (F_2 \vec{\otimes} I_2)\right)\left(L^4_2 \vec{\otimes} I_2\right)$$

∑-OL:

$$\left(\sum_{j=0}^{1}\left(S_{\imath_2\otimes(j)_2} F_2 \mathrm{Map}^2_{x\mapsto\omega_4^{2i+j}x} G_{\imath_2\otimes(j)_2}\right)\sum_{j=0}^{1}\left(S_{(j)_2\otimes\imath_2} F_2 G_{\imath_2\otimes(j)_2}\right)\right)\vec{\otimes} I_2 \ldots$$

C Code:

```
void dft8(_Complex double *Y, _Complex double *X) {
    __m256d s38, s39, s40, s41,...
    __m256d  *a17, *a18;
    a17 = ((__m256d  *) X);
    s38 = *(a17);
    s39 = *((a17 + 2));
    t38 = _mm256_add_pd(s38, s39);
    t39 = _mm256_sub_pd(s38, s39);
    ...
    s52 = _mm256_sub_pd(s45, s50);
    *((a18 + 3)) = s52;
}
```



Internal Representation: SIMT Σ-SPL

```
... * SIMTSUM( ASIMTLoopDim(),
   SIMTISum(ASIMTLoopDim(), i81, 3,
     SIMTISum(ASIMTLoopDim(), i215, 13, BB(
       Scat(fTensor(fCompose(H(130, 78, 0, 1), fTensor(fBase(i81), fId(2), fBase(i215))), fId(2))) *
       Blk([ [ V(1), V(0), V(1), V(0) ], [ V(0), V(1), V(0), V(1) ],
             [ V(1), V(0), V(-1), V(0) ], [ V(0), V(1), V(0), V(-1) ] ]) *
       RCDiag(FDataOfs(D62, 4, mul(V(4), i215)), I(2)) * Gath(fTensor(fId(2), fBase(i215), fId(2))) )
     ) *
     SIMTSUM( ASIMTLoopDim(),
       SIMTSUM( ASIMTLoopDim(),
         ISumAcc(i445, 13, BB(
           ScatAcc(fTensor(fCompose(H(26, 13, 0, 1), fBase(13, V(0))), fId(2))) *
           Blk([ [ V(1), V(0) ], [ V(0), V(1) ] ]) ) *
           COND(eq(i445, V(0)),
             BB(Gath(fTensor(fAdd(13, 1, 0), fId(2)))),
             BB(Gath(fTensor(fAdd(13, 1, i445), fId(2))))
           ) ),
         SIMTISum(ASIMTLoopDim(), i431, 6,
           ISumAcc(i433, 7, BB(
             ScatAcc(fTensor(fCompose(H(26, 13, 0, 1), Lambda([ i437 ], nth(D38, i437)).setRange(13), H(13, 12, 1, 1),
               fTensor(fId(2), fBase(i431))), fId(2))) *
               Blk([ [ V(1), V(0), V(1), V(0) ], [ V(0), V(1), V(0), V(1) ],
                     [ V(1), V(0), V(-1), V(0) ], [ V(0), V(1), V(0), V(-1) ] ]) ) *
             COND(eq(i433, V(0)),
               BB(Blk([ [ V(1), V(0) ], [ V(0), V(1) ], [ V(0), V(0) ], [ V(0), V(0) ] ]) *
                 Gath(fTensor(fCompose(Lambda([ i440 ], nth(D39, i440)).setRange(13), fAdd(13, 1, 0)), fId(2)))),
               BB(SIMTSUM( ASIMTLoopDim(),
                   Scat(fTensor(fBase(2, 0), fId(2))) *
                   Blk([[nth(D37, add(mul(V(6), i431), sub(i433, V(1)))), V(0.0) ],
                       [ V(0.0), nth(D37, add(mul(V(6), i431), sub(i433, V(1)))) ] ]) *
                   Gath(fTensor(fBase(2, 0), fId(2))),
                   Scat(fTensor(fBase(2, 1), fId(2))) *
                   RCDiag(RCData(fConst(TReal, 1, E(4))), I(2)) *
                   Blk([ [ nth(D37, add(V(36), mul(V(6), i431), sub(i433, V(1)))), V(0.0) ],
                         [ V(0.0), nth(D37, add(V(36), mul(V(6), i431), sub(i433, V(1)))) ] ]) *
                   Gath(fTensor(fBase(2, 1), fId(2)))) *
                 Blk([ [ V(1), V(0), V(1), V(0) ], [ V(0), V(1), V(0), V(1) ],
                       [ V(1), V(0), V(-1), V(0) ], [ V(0), V(1), V(0), V(-1) ] ]) *
                 Gath(fTensor(fCompose(Lambda([ i440 ], nth(D39, i440)).setRange(13), fAdd(13, 12, 1),
                      fTensor(fId(2), fBase(6, sub(i433, V(1))))), fId(2)))
           ) ) ) ) ) * ...
```

Internal Representation: CUDA iCode

```
func(TVoid, "transform", [ hY, hX ],
    chain(
        decl([ b664, g1 ],
            chain(
                cu_memcpy(X, hX, 35937, cudaMemcpyHostToDevice),
                    decl([ ms1, start1, stop1 ],
                        chain(
                            cu_event_create(addrof(start1)),
                            cu_event_create(addrof(stop1)),
                            cu_check_errors(cu_event_record(start1, V(false))),
                            ker_code0<<<g1,b664>>>(X, Y),
                            cu_check_errors(cu_event_record(stop1, V(false))),
                            cu_check_errors(cu_event_synchronize(stop1, V(false))),
                            assign(ms1, V(0)),
                            cu_event_elapsed_time(addrof(ms1), start1, stop1),
                            cu_event_destroy(start1),
                            cu_event_destroy(stop1)
                         )
                     ),
                     cu_memcpy(hY, Y, 884736, cudaMemcpyDeviceToHost)
                  )
               )
            )
         ),
         func(TVoid, "destroy", Set([  ]),
            chain(
               cu_free(Y),
               cu_free(X)
            )
         )
```

SPIRAL Generated CUDA Code

```
__global__ void ker_code0(int *D48, double *D49, double *D50, double *D51, int *D52, double  *X) {
    __shared__ double T235[260];
    ...
    if (((threadIdx.x < 13))) {
        for(int i96 = 0; i96 <= 4; i96++) {
            int a31, a32, a33, a34;
            a31 = (2*i96);
            a32 = (threadIdx.x + (13*a31));
            a33 = (threadIdx.x + (13*((a31 + 5) % 10)));
            a34 = (4*i96);
            *((((T235 + 0) + a34) + (20*threadIdx.x))) = (*((T6 + a32)) + *((T6 + a33)));
            *(((1 + (T235 + 0) + a34) + (20*threadIdx.x))) = 0.0;
            *(((2 + (T235 + 0) + a34) + (20*threadIdx.x))) = (*((T6 + a32)) - *((T6 + a33)));
            *(((3 + (T235 + 0) + a34) + (20*threadIdx.x))) = 0.0;
        }
        double t261, t262, t263, t264, t265, t266, t267, t268;
        int a129;
        t263 = (*((((T235+0)+12)+(20*threadIdx.x)))+*((((T235+0)+8)+(20*threadIdx.x))));
        t264 = (*((((T235+0)+12)+(20*threadIdx.x)))-*((((T235+0)+8)+(20*threadIdx.x))));
        ...
        *((3 + T5 + a129)) = ((0.58778525229247314*t268) - (0.95105651629515353*t266));
    }
    __syncwarp();
    if (((threadIdx.x < 1))) {
        double t305, t306, t307, t308, t309, t310, t311, t312, t313, t314, t315, t316;
        int a387;
        t305 = (*((T5 + 12)) + *((T5 + 144)));
        ...
```

3,000 lines of code, kernel fusion, cross call data layout transforms