Formula Level Loop Optimization

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Contents

1 Overview

1.1 Basic Objects

1.1.1 Integer Expressions

Objects: $m, n \in \mathbb{N}_0$
Symbols: $0, 1, \ldots$
Operators: $m+n$, $m-n$, mn, m^n, $m \bmod n$, $\left\lfloor \frac{m}{n} \right\rfloor$

1.1.2 Real and Complex Expressions

Objects: $c, d \in \mathbb{C}$, $\alpha \in \mathbb{R}$
Symbols: π, i, e, ω_n
Operators: $c+d$, cd, c^k, $\sin\alpha$, $\cos\alpha$, $\Re(c)$, $\Im(c)$, for $k \in \mathbb{N}_0$

1.1.3 Integer Sets

Objects: $M, N \subset \mathbb{N}_0$; $|M|, |N| \in \mathbb{N}_0$
Symbols: $\emptyset$, $\mathbb{I}_{l,r}$
Operators: $M \cup N$, $M \cap N$, $k+N$, kN, for $k \in \mathbb{N}_0$

1.1.4 Interval Mapping Functions

Objects: $f^{m\to M} \in \mathbb{I}_{0,M}^{\mathbb{I}_{0,m}}, g^{n\to N}, h_j^{n\to N} \in \mathbb{I}_{0,N}^{\mathbb{I}_{0,n}}$, $j \in \mathbb{I}_{0,n}$
Symbols: p_n, $(j)_n$, $\mathrm{x}_n^{j,t}$, $(m)_+^{n\to N}$, $(m)_\times^{n\to N}$
Operators: $f \oplus g$, $f \otimes g$, $\begin{bmatrix} f^{m\to M} \\ g^{n\to M} \end{bmatrix}$, $f^{M\to N} \circ g^{m\to M}$, $\underbrace{h_{(.)}^{n\to N}}_{.}$, $\left\langle f^{m\to M} | g^{m\to M} \right\rangle_k$, for $k \in \mathbb{N}_0$

1.1.5 Permutation Generating Functions

Objects: $\pi^{m\circlearrowleft} \in \mathrm{S}_m \subset \mathbb{I}_{0,m}^{\mathbb{I}_{0,m}}, \sigma^{n\circlearrowleft} \in \mathrm{S}_n \subset \mathbb{I}_{0,n}^{\mathbb{I}_{0,n}}$
Symbols: $\imath_n$, $\jmath_n$, z_n^k, ℓ_m^{mn}, $\alpha_{a,b}^n$, $\kappa_{a,b}^n$
Operators: π^{-1}, $\pi \oplus \sigma$, $\pi \otimes \sigma$, $\pi^{n\circlearrowleft} \circ \sigma^{n\circlearrowleft}$, $\langle \pi^{n\circlearrowleft} | \sigma^{n\circlearrowleft} \rangle_k$, for $k \in \mathbb{N}_0$

1.1.6 One-dimensional Matrix Generating Functions

Objects: $f^{n\to\mathbb{C}}$, $g^{n\to\mathbb{C}}$
Symbols: $\mathrm{o}^{n\to\mathbb{C}}$, $\imath^{n\to\mathbb{C}}$, $\pm\imath^{n\to\mathbb{C}}$, $(c)^{n\to\mathbb{C}}$, δ_N^n, for $c \in \mathbb{C}$, $N \subset \mathbb{I}_{0,n}$
Operators: $f+g$, $f\,g$, $\frac{1}{f}$, $f \oplus g$, $f \otimes g$, $f^{n\to\mathbb{C}} \circ h^{n'\to n}$, $\left\langle f^{n\to\mathbb{C}} | g^{n\to\mathbb{C}} \right\rangle_k$, for $k \in \mathbb{N}_0$

1.1.7 Σ-SPL Formulas

Objects:	$A, B \in \mathbb{C}^{m\times n}$
Symbols:	$0_{m\times n}$, I_n, J_n, L^{mn}_m, T^{mn}_n, R_α ...
Function Operators:	$\mathrm{diag}\left(f^{n\to\mathbb{C}}\right)$, $\mathrm{col}\left(f^{n\to\mathbb{C}}\right)$, $\mathrm{row}\left(f^{n\to\mathbb{C}}\right)$, $\mathrm{circ}\left(f^{n\to\mathbb{C}}\right)$, $\mathrm{scirc}\left(f^{n\to\mathbb{C}}\right)$,
	$\mathrm{toepl}\left(f^{n\to\mathbb{C}}\right)$, $\mathrm{perm}\left(\pi^{n\circlearrowleft}\right)$, $\mathrm{mon}\left(\pi^{n\circlearrowleft}, f^{n\to\mathbb{C}}\right)$
Σ-Operators:	$A+B$, $A +_{\mathrm{acc}} B$, $\sum_{i=0}^{n-1} A_i$, $\sum_{i=0}^{n-1}{}_{\mathrm{acc}}\, A_i$,
SPL-Operators	$A\oplus B$, $A\otimes B$, $A\otimes_k B$, $A\otimes^k B$, $A\,B$, $\prod_{i=0}^{n-1} A_i$, $\bigoplus_{i=0}^{n-1} A_i$,
	$[A\vert B]$, $\left[\,\Big\vert_{i=0}^{n-1}\right] A_i$, $\left[\begin{smallmatrix}A\\B\end{smallmatrix}\right]$, $\left[\,\overline{}_{i=0}^{n-1}\right] A_i$, $\mathrm{blockmat}_{i,j}\left(A_{i,j}\right)$

2 Definitions

2.1 Integer Sets

Definition 1 (Integer Interval)**.**

$$\mathbb{I}_n = \{0\ldots, n-1\}$$

Definition 2 (Integer Interval)**.**

$$\mathbb{I}_{m,n} = \{m\ldots, n-1\}$$

Definition 3 (Integer Interval)**.**

$$k + \{n_0, \ldots, n_m\} = \{k+n_0, \ldots, k+n_m\}$$

Definition 4 (Integer Interval)**.**

$$k\{n_0, \ldots, n_m\} = \{kn_0, \ldots, kn_m\}$$

2.2 General Functions

Notation 2.1. A function

$$f : \begin{cases} D \to R \\ i \mapsto f(i) \end{cases}$$

is denoted by

$$f^{D\to R}.$$

Definition 5 (Picture of a Set)**.** The picture of a set

$$I' \subseteq I$$

under a functions

$$f : \begin{cases} I \to J \\ i \mapsto f(i) \end{cases}$$

is defined as

$$f(I') = \left\{f(i)\right\}_{i\in I'}.$$

Definition 6 (Picture of a Parameterized Function Under a Set of Parameters)**.** For

$$f_j : \begin{cases} I \to J \\ i \mapsto f_j(i) \end{cases}$$

The picture under a set of parameters $P = \{p_0, \ldots, p_k\}$ is given by

$$f_P(i) = \left\{f_j(i)\right\}_{j\in P}.$$

Definition 7 (Concatenation)**.**

$$g^{J\to K}\circ f^{I\to J}:\begin{cases} I\to K\\ i\mapsto g(f(i))\end{cases}$$

Definition 8 (Pseudo Inversion of Injective Functions)**.** For an injective function f with

$$f:\begin{cases} I\to J\\ i\mapsto f(i)\end{cases},$$

the pseudo inverse f^{-1} is defined by

$$f^{-1}:\begin{cases} f(I)\to I\\ i\mapsto j \quad \text{with } f(j)=i\end{cases}$$

Definition 9 (Binding a Function Parameter to the Variable)**.** For a parameterized function

$$f_j$$

the parameter is set as the actual argument of the function by

$$\underbrace{f_{(\cdot)}}_{\cdot}:\begin{cases} D\to R\\ i\mapsto f_i(i).\end{cases}$$

2.3 Interval Mapping Functions

2.3.1 Definitions

Definition 10 (Interval Mapping Function)**.** A function of form

$$f:\begin{cases} \mathbb{I}_n\to \mathbb{I}_N\\ i\mapsto f(i)\end{cases}$$

is called interval mapping function.

Notation 2.2. An interval mapping function

$$f:\begin{cases} \mathbb{I}_n\to \mathbb{I}_N\\ i\mapsto f(i)\end{cases}$$

is denoted by

$$f^{n\to N}.$$

Notation 2.3. An unnamed interval mapping function

$$\begin{cases} \mathbb{I}_n\to \mathbb{I}_N\\ i\mapsto f(i)\end{cases}$$

is denoted by

$$n\to N: i\mapsto f(i).$$

Definition 11 (Projection Interval Mapping Function)**.** Projection interval mapping functions are given by

$$\mathrm{p}_n:\begin{cases} \mathbb{I}_n\to \mathbb{I}_1\\ i\mapsto 0\end{cases}$$

Definition 12 (Basis Interval Mapping Function)**.** Basis-n interval mapping functions are given by

$$(j)_n : \begin{cases} \mathbb{I}_1 \to \mathbb{I}_n \\ i \mapsto j \end{cases} \qquad \text{with} \quad 0 \le j < n.$$

Notation 2.4. If the value of n is clear from the context (e. g., j is the index of a iterative construct, and $0 \le j < n$), the shortcut

$$j := (j)_n \quad \text{with} \quad 0 \le j < n.$$

is used.

Definition 13 (K/M Basis Interval Mapping Function)**.**

$$\mathrm{x}_n^{j,t} : \begin{cases} \mathbb{I}_1 \to \mathbb{I}_n \\ i \mapsto j + (t \bmod 2)\big((n-1) - 2j\big) \end{cases} \qquad , \quad 0 \le j < n$$

Property 2.1.

$$\big(\langle \imath_n | \jmath_n \rangle_t (j)\big)_n = \mathrm{x}_n^{j,t}$$

Property 2.2.

$$\langle \imath_n | \jmath_n \rangle_t \circ (j)_n = \mathrm{x}_n^{j,t}$$

Definition 14 (Z Basis Interval Mapping Function)**.**

$$\mathrm{s}_n^{j,k} : \begin{cases} \mathbb{I}_1 \to \mathbb{I}_n \\ i \mapsto j + k \bmod n \end{cases} \qquad , \quad 0 \le j < n$$

Property 2.3.

$$\big(\, \mathrm{z}_n^k(j)\big)_n = \mathrm{s}_n^{j,k}$$

Property 2.4.

$$\mathrm{z}_n^k \circ (j)_n = \mathrm{s}_n^{j,k}$$

Definition 15 (Integer Add Interval Mapping Function)**.** Add-m interval mapping functions are given by

$$(m)_+^{n \to N} : \begin{cases} \mathbb{I}_n \to \mathbb{I}_N \\ i \mapsto i + m \end{cases} \qquad , \quad m \in \mathbb{N}_0 \quad , \quad N \ge m + n.$$

Definition 16 (Integer Multiply Interval Mapping Function)**.** Multiply-m interval mapping functions are given by

$$(m)_\times^{n \to N} : \begin{cases} \mathbb{I}_n \to \mathbb{I}_N \\ i \mapsto im \end{cases} \qquad , \quad m \in \mathbb{N} \quad , \quad N \ge m(n-1) + 1.$$

Property 2.5 (Changing the Range of a Interval Mapping Function)**.** For an interval mapping function

$$f^{n \to N}$$

the range is extended to $M \ge N$ by

$$f^{n \to M} := (0)_+^{N \to M} \circ f^{n \to N}.$$

Definition 17 (Rader Interval Mapping Function)**.**

$$w_{\varphi,g}^{n \to N} : \begin{cases} \mathbb{I}_n \to \mathbb{I}_N \\ i \mapsto \varphi g^i \bmod N \end{cases} \qquad , \quad N \quad \text{prime, } g \text{ generator of } \mathbb{Z}_N^\times.$$

2.3.2 Operators

Definition 18 (Direct Sum of Interval Mapping Functions)**.**

$$f^{m\to M} \oplus g^{n\to N} : \begin{cases} \mathbb{I}_{m+n} \to \mathbb{I}_{M+N} \\ i \mapsto \begin{cases} f(i) & \text{if } 0 \le i < m \\ g(i-m) + M & \text{if } m \le i < m+n \end{cases} \end{cases}$$

Definition 19 (Tensor Product of Interval Mapping Functions)**.**

$$f^{m\to M} \otimes g^{n\to N} : \begin{cases} \mathbb{I}_{mn} \to \mathbb{I}_{MN} \\ i \mapsto Nf\left(\lfloor \frac{i}{n} \rfloor\right) + g\big(i \bmod n\big) \end{cases}$$

Definition 20 (Gamma Product of Interval Mapping Functions)**.**

$$f^{m\to M} \boxtimes g^{n\to N} : \begin{cases} \mathbb{I}_{mn} \to \mathbb{I}_{MN} \\ i \mapsto Nf\left(\lfloor \frac{i}{n} \rfloor\right) + Mg\big(i \bmod n\big) \bmod MN \end{cases} \qquad \text{for} \quad \gcd(M,N) = 1$$

Definition 21 (Stacking of Interval Mapping Functions)**.**

$$\begin{bmatrix} f^{n_0\to N} \\ g^{n_1\to N} \end{bmatrix} : \begin{cases} \mathbb{I}_{n_0+n_1} \to \mathbb{I}_N \\ i \mapsto \begin{cases} f(i) & \text{if } 0 \le i < n_0 \\ g(i-n_0) & \text{if } n_0 \le i < n_0 + n_1 \end{cases} \end{cases}$$

Definition 22 (Alternator of Interval Mapping Function)**.**

$$\left\langle f^{n\to N} \middle| g^{n\to N} \right\rangle_m : \begin{cases} \mathbb{I}_n \to \mathbb{I}_N \\ i \mapsto f(i) + (m \bmod 2)\big(g(i) - f(i)\big) \end{cases}$$

2.4 Permutation Generating Functions

Definition 23 (Permutation Generating Function)**.** Permutation generating functions are bijective interval mapping functions of type

$$\pi : \begin{cases} \{0,\dots,n-1\} \to \{0,\dots,n-1\} \\ i \mapsto \pi(i) \end{cases}$$

with the permutation

$$\pi \in \mathrm{S}_n\,.$$

Notation 2.5. A permutation generating function

$$\pi : \begin{cases} \{0,\dots,n-1\} \to \{0,\dots,n-1\} \\ i \mapsto \pi(i) \end{cases} \qquad , \quad \pi \in \mathrm{S}_n$$

is denoted by

$$\pi^{n\circlearrowleft}.$$

2.4.1 Definitions

Definition 24 (Identity Permutation Generating Function)**.** The identity permutation generating function is given by

$$\imath_n : \begin{cases} \{0,\dots,n-1\} \to \{0,\dots,n-1\} \\ i \mapsto i \end{cases} \qquad .$$

Definition 25 (Opposite Diagonal Permutation Generating Function)**.** The opposite diagonal permutation

$$\mathrm{J}_n = \mathrm{perm}\,(\jmath_n)$$

is generated by

$$\jmath_n : \begin{cases} \{0,\dots,n-1\} \to \{0,\dots,n-1\} \\ i \mapsto n-1-i \end{cases}.$$

Definition 26 (Cyclic Shift Permutation Generating Function)**.** The cyclic shift permutation

$$\mathrm{Z}_n^k = \mathrm{perm}\left(\mathrm{z}_n^k\right)$$

is generated by

$$\mathrm{z}_n^k : \begin{cases} \{0,\dots,n-1\} \to \{0,\dots,n-1\} \\ i \mapsto (i+k) \bmod n \end{cases}.$$

Definition 27 (Stride Permutation Generating Function)**.** The stride permutation

$$\mathrm{L}_m^{mn} = \mathrm{perm}\,(\ell_m^{mn})$$

is generated by

$$\ell_m^{mn} : \begin{cases} \{0,\dots,mn-1\} \to \{0,\dots,mn-1\} \\ i \mapsto \begin{cases} (im) \bmod (mn-1) & \text{if } i < mn-1 \\ mn-1 & \text{if } i = mn-1 \end{cases} \end{cases}.$$

Property 2.6 (Stride Permutation Generating Function)**.** The stride permutation

$$\mathrm{L}_m^{mn} = \mathrm{perm}\,(\ell_m^{mn})$$

is generated by

$$\ell_m^{mn} : \begin{cases} \{0,\dots,mn-1\} \to \{0,\dots,mn-1\} \\ i \mapsto \left\lfloor \frac{i}{n} \right\rfloor + m(i \bmod n) \end{cases}.$$

Definition 28 (Odd Stride Permutation Generating Function)**.** The odd stride permutation

$$\mathrm{L}_r^n = \mathrm{perm}\,(\ell_r^n)$$

is generated by

$$\overline{\ell}_r^n : \begin{cases} \{0,\dots,n-1\} \to \{0,\dots,n-1\} \\ i \mapsto (ir) \bmod (n-1) \end{cases} \qquad \text{for} \quad \gcd(n,r) = 1.$$

Definition 29 (V Permutation Generating Function)**.** The permutation

$$\mathrm{V}_m^{mn} = \underbrace{\mathrm{I}_n \oplus \mathrm{J}_n \oplus \cdots}_{m \text{ summands}}$$

is generated by

$$\mathrm{v}_m^{mn} = \underbrace{\imath_n \oplus \jmath_n \oplus \cdots}_{m \text{ summands}}.$$

Definition 30 (K Permutation Generating Function)**.** The permutation

$$\mathrm{K}_m^{mn} = (\mathrm{I}_n \oplus \mathrm{J}_n \oplus \cdots)\,\mathrm{L}_m^{mn}$$

is generated by

$$\mathrm{k}_m^{mn} = \ell_m^{mn} \circ \left(\imath_n \oplus \jmath_n \oplus \cdots\right).$$

Definition 31 (M Permutation Generating Function)**.** The permutation

$$\mathrm{M}_m^{mn} = \mathrm{L}_m^{mn}(\mathrm{I}_m \oplus \mathrm{J}_m \oplus \cdots)$$

is generated by

$$\mathrm{m}_m^{mn} = \left(\imath_m \oplus \jmath_m \oplus \cdots\right) \circ \ell_m^{mn}.$$

Property 2.7.

$$\mathrm{m}_m^{mn} : \begin{cases} \{0,\ldots,mn-1\} \to \{0,\ldots,mn-1\} \\ i \mapsto \lfloor \frac{i}{n} \rfloor + m(i \bmod n) + \left((i \bmod n) \bmod 2\right)\left((m-1) - 2\lfloor \frac{i}{n} \rfloor\right) \end{cases}$$

Definition 32 (Affine Permutation Generating Function)**.** The affine permutation

$$\mathrm{A}_{a,b}^n = \mathrm{perm}\left(\alpha_{a,b}^n\right) \quad , \quad (a,n) = 1$$

is generated by

$$\alpha_{a,b}^n : \begin{cases} \{0,\ldots,n-1\} \to \{0,\ldots,n-1\} \\ i \mapsto ai + b \bmod n \end{cases} \quad , \quad a \nmid n.$$

Property 2.8 (Affine Permutation Generating Function)**.** For

$$a \mid n+1,$$

the affine permutation

$$\mathrm{A}_a^n := \mathrm{A}_{a,0}^n \quad , \qquad a \nmid n$$

is generated by

$$\alpha_a^n : \begin{cases} \{0,\ldots,n-1\} \to \{0,\ldots,n-1\} \\ i \mapsto \left\lfloor \frac{ai}{n+1} \right\rfloor + a(i \bmod \frac{n+1}{a}) \end{cases} .$$

Definition 33 (Multiplicative Permutation Generating Function)**.** The multiplicative permutation

$$\mathrm{K}_{a,b}^n = \mathrm{perm}\left(\kappa_{a,b}^n\right) \quad , \quad n \text{ prime and } a \text{ primitive element in } \mathbb{Z}_n,$$

is generated by

$$\kappa_{a,b}^n : \begin{cases} \{0,\ldots,n-2\} \to \{0,\ldots,n-2\} \\ i \mapsto (ba^i \bmod n) - 1 \end{cases} .$$

Definition 34 (CRT Permutation Generating Function)**.**

$$V_{\alpha,\beta}^{r,s} : \begin{cases} \{0,\ldots,rs-1\} \to \{0,\ldots,rs-1\} \\ i \mapsto \lfloor \frac{i}{s} \rfloor \alpha s + (i \bmod s)\beta r \bmod rs \end{cases}$$

Property 2.9 (CRT Permutation Generating Function)**.**

$$\Gamma^{rs} := \mathrm{perm}\left(V_{e_r,e_s}^{r,s}\right)$$

with

$$\begin{aligned} e_r \bmod s &= 0 \\ e_r \bmod r &= 1 \\ e_s \bmod s &= 1 \\ e_s \bmod s &= 0 \end{aligned}$$

Property 2.10 (CRT Permutation Generating Function)**.**

$$V_{\alpha,\beta}^{r,s} = \overline{\ell}_\alpha^r \boxtimes \overline{\ell}_\beta^s$$

Property 2.11.

$$V_{1,1}^{r,s} = \imath_r \boxtimes \imath_s$$

2.4.2 Operators

Definition 35 (Inverse of Permutation Generating Function)**.** The inverse of a permutation generating function

$$\pi : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, n-1\} \\ i \mapsto \pi(i) \end{cases} \quad , \quad \pi \in \mathrm{S}_n \, .$$

is given by

$$\pi^{-1} : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, n-1\} \\ i \mapsto j \quad \text{with } \pi(j) = i \end{cases} \quad , \quad \pi^{-1} \in \mathrm{S}_n \, .$$

Definition 36 (Conjugation of Permutation Generating Function)**.**

$$\pi^\sigma = \sigma \circ \pi \circ \sigma^{-1}$$

Definition 37 (Alternator of Permutation Generating Function)**.**

$$\left\langle \pi^{n\circlearrowleft} | \sigma^{n\circlearrowleft} \right\rangle_m : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, n-1\} \\ i \mapsto \pi(i) + (m \bmod 2)\big(\sigma(i) - \pi(i)\big) \end{cases}$$

2.5 Matrix Generating Functions

2.5.1 Definitions

Definition 38 (Scalar Matrix Generating Function)**.** Scalar matrix generating functions are of type

$$f : \begin{cases} \{0, \ldots, n-1\} \to \mathbb{C} \\ i \mapsto f(i) \end{cases} \quad .$$

Notation 2.6. A scalar matrix generating function

$$f : \begin{cases} \{0, \ldots, n-1\} \to \mathbb{C} \\ i \mapsto f(i) \end{cases}$$

is denoted by

$$f^{n \to \mathbb{C}}.$$

Notation 2.7. An unnamed scalar matrix generating function

$$\begin{cases} \{0, \ldots, n-1\} \to \mathbb{C} \\ i \mapsto f(i) \end{cases}$$

is denoted by

$$n \to \mathbb{C} : i \mapsto f(i).$$

Definition 39 (Zero Function)**.**

$$\mathrm{o}^{n \to \mathbb{C}} : \begin{cases} \{0, ..., n-1\} \to \mathbb{C} \\ i \mapsto 0 \end{cases}$$

Definition 40 (One Function)**.**

$$\imath^{n \to \mathbb{C}} : \begin{cases} \{0, ..., n-1\} \to \mathbb{C} \\ i \mapsto 1 \end{cases}$$

Definition 41 (Alternate Sign Function)**.**

$$\pm\imath^{n \to \mathbb{C}} : \begin{cases} \{0, ..., n-1\} \to \mathbb{C} \\ i \mapsto (-1)^i \end{cases}$$

Definition 42 (Constant Function)**.**

$$(c)^{n\to\mathbb{C}} : \begin{cases} \{0, ..., n-1\} \to \mathbb{C} \\ i \mapsto c \end{cases} \quad , \quad c \in \mathbb{C}$$

Definition 43 (Delta Function)**.**

$$\delta_N^n : \begin{cases} \{0, ..., n-1\} \to \mathbb{C} \\ i \mapsto \begin{cases} 1 & \text{if } i \in N \\ 0 & \text{else} \end{cases} \end{cases} \quad , \quad 0 \leq n \quad , \quad N \subseteq \{0, \ldots, n-1\}$$

Definition 44 (Twiddle Generating Function)**.**

$$\mathrm{t}_n^{mn} : \begin{cases} \{0, \ldots, mn-1\} \to \mathbb{C} \\ i \mapsto \omega_{mn}^{(i \bmod n)\lfloor \frac{i}{n} \rfloor} \end{cases} \quad , \quad \omega_n = \sqrt[n]{i}$$

2.5.2 Operators

Definition 45 (Sum of Scalar Generating Functions)**.** The sum of the two scalar mapping functions

$$f : \begin{cases} \{0, \ldots, n-1\} \to \mathbb{C} \\ i \mapsto f(i) \end{cases} \quad \text{and} \quad g : \begin{cases} \{0, \ldots, n-1\} \to \mathbb{C} \\ i \mapsto g(i) \end{cases}$$

is given by the scalar generating function

$$f + g : \begin{cases} \{0, \ldots, n-1\} \to \mathbb{C} \\ i \mapsto f(i) + g(i) \end{cases} \quad .$$

Definition 46 (Direct Sum of Scalar Generating Functions)**.** The direct sum of the two scalar generating functions

$$f : \begin{cases} \{0, \ldots, m-1\} \to \mathbb{C} \\ i \mapsto f(i) \end{cases} \quad \text{and} \quad g : \begin{cases} \{0, \ldots, n-1\} \to \mathbb{C} \\ i \mapsto g(i) \end{cases}$$

is given by the scalar generation function

$$f \oplus g : \begin{cases} \{0, \ldots, m+n-1\} \to \mathbb{C} \\ i \mapsto (f \oplus g)(i) \end{cases}$$

with

$$(f \oplus g)(i) = \begin{cases} f(i) & \text{if } i \in \{0, \ldots, m-1\} \\ g(i-m) & \text{if } i \in \{m, \ldots, n-1\} \end{cases} .$$

Definition 47 (Product of Scalar Generating Functions)**.** The product of the two scalar generating functions

$$f : \begin{cases} \{0, \ldots, n-1\} \to \mathbb{C} \\ i \mapsto f(i) \end{cases} \quad \text{and} \quad g : \begin{cases} \{0, \ldots, n-1\} \to \mathbb{C} \\ i \mapsto g(i) \end{cases}$$

is given by the scalar generation function

$$fg : \begin{cases} \{0, \ldots, n-1\} \to \mathbb{C} \\ i \mapsto f(i)g(i) \end{cases} \quad .$$

Definition 48 (Tensor Product of Scalar Generating Functions)**.** The tensor product of the two scalar generating functions

$$f : \begin{cases} \{0, \ldots, m-1\} \to \mathbb{C} \\ i \mapsto f(i) \end{cases} \qquad \text{and} \quad g : \begin{cases} \{0, \ldots, n-1\} \to \mathbb{C} \\ i \mapsto g(i) \end{cases}$$

is given by the scalar generation function

$$f \otimes g : \begin{cases} \{0, \ldots, mn-1\} \to \mathbb{C} \\ (i, j) \mapsto f\left(\left\lfloor \frac{i}{n} \right\rfloor\right) g\big(i \bmod n\big) \end{cases}.$$

Definition 49 (Alternator of Scalar Generating Function)**.**

$$\left\langle f^{n \to \mathbb{C}} | g^{n \to \mathbb{C}} \right\rangle_m : \begin{cases} \{0, \ldots, n-1\} \to \mathbb{C} \\ i \mapsto f(i) + (m \bmod 2)\big(g(i) - f(i)\big) \end{cases}$$

2.6 Parameterized Matrices

Definition 50 (Standard Basis)**.** Let $e_0^n, e_1^n, \ldots, e_{n-1}^n$ denote the vectors in $\mathbb{C}^{n \times 1}$ with a 1 in the component given by the subscript and 0 elsewhere. The set

$$B_n = \{e_i^n : i = 0, 1, \ldots, n-1\}$$

is the standard basis of $\mathbb{C}^{n \times 1}$.

2.6.1 Matrices Parameterized by Interval Mapping Functions

Definition 51 (Gather Matrix)**.** The interval mapping function

$$f : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, N-1\} \\ i \mapsto f(i) \end{cases}$$

generates the gather matrix

$$\mathrm{G}_{f^{n \to N}} := \underset{i=0}{\overset{n-1}{[\!-\!]}} \left(e_{f(i)}^N\right)^\top.$$

Notation 2.8.

$$\mathrm{G}_{f^{n \to N}} = \mathrm{G}_f^{N,n}$$

Definition 52 (Scatter Matrix)**.** The interval mapping function

$$f : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, N-1\} \\ i \mapsto f(i) \end{cases}$$

generates the scatter matrix

$$\mathrm{S}_{f^{n \to N}} := \underset{i=0}{\overset{n-1}{[\,|\,]}} e_{f(i)}^N.$$

Notation 2.9.

$$\mathrm{S}_{f^{n \to N}} = \mathrm{S}_f^{N,n}$$

Definition 53 (Parameterized Permutation)**.** The permutation generation function

$$\pi : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, n-1\} \\ i \mapsto \pi(i) \end{cases} \qquad \text{with} \quad \pi \in \mathrm{S}_n$$

generates a permutation matrix

$$\mathrm{perm}\left(\pi^{n\circlearrowleft}\right) := \underset{i=0}{\overset{n-1}{[\!-\!]}} \left(e_{\pi(i)}^N\right)^\top.$$

2.6.2 Matrices Parameterized by Integers

Property 2.12 (Matrices Generated by Scalar Functions)**.** A scalar matrix generation functions

$$f : \begin{cases} \{0, \dots, n-1\} \to \mathbb{C} \\ i \mapsto f(i) \end{cases} .$$

can generate the matrices

$$\operatorname{diag}(f) \quad , \quad \operatorname{row}(f) \quad , \quad \text{and} \quad \operatorname{column}(f).$$

Definition 54 (Integer Parameterized Matrices)**.** Integer parameterized matrices are of type

$$A : \begin{cases} \{0, \dots, M-1\} \to \mathbb{C}^{m \times n} \\ i \mapsto \left[a_{j,k}(i)\right]_{\substack{0 \le j < m \\ 0 \le k < n}} \end{cases}$$

with the family

$$\left\{a_{j,k}\right\}_{\substack{0 \le j < m \\ 0 \le k < n}}$$

of scalar matrix generation functions.

Notation 2.10. For integer parameterized matrices the notation

$$A_i := A(i)$$

is used.

Property 2.13 (Matrices Generated by Integer Parameterized Matrices)**.** Integer parameterized matrices are used to generate

$$\sum_{i=0}^{m-1} A_i \qquad \prod_{i=0}^{m-1} A_i \qquad \bigoplus_{i=0}^{m-1} A_i \qquad \bigotimes_{i=0}^{m-1} A_i$$

$$\bigoplus_{i=0}^{m-1}{}_k A_i \qquad \bigoplus_{i=0}^{m-1}{}^k A_i \qquad \underset{i=0}{\overset{m-1}{\left[\,\text{---}\,\right]}} A_i \qquad \underset{i=0}{\overset{m-1}{\left[\;\middle|\;\right]}} A_i$$

2.6.3 Monomial Matrices

Definition 55 (Monomial Matrix)**.** A (not necessarily invertible) monomial matrix is given by

$$\operatorname{mon}\left(\pi^{n\circlearrowleft}, f^{n\to\mathbb{C}}\right) = \operatorname{perm}\left(\pi^{n\circlearrowleft}\right) \operatorname{diag}\left(f^{n\to\mathbb{C}}\right) \quad , \quad \pi \in \mathrm{S}_n .$$

2.7 Extensions

2.7.1 Extension Matrices

Definition 56 (Zero Extension)**.**

$$\mathrm{E}^{\mathrm{zero}}_{n,l,r} = \mathrm{S}_{(l)_+^{n \to n+l+r}}$$

Definition 57 (Index Extension)**.**

$$\mathrm{E}^{\mathrm{idx}}_{n,\mathrm{l}^{l\to n},\mathrm{r}^{r\to n}} = \mathrm{G}_{\begin{bmatrix} \mathrm{l}^{l\to n} \\ \imath_n \\ \mathrm{r}^{r\to n} \end{bmatrix}}$$

Definition 58 (Scaled Index Extension)**.**

$$\mathrm{E}^{\mathrm{sidx}}_{n,(\mathrm{l}_i^{l\to n},\mathrm{l}_s^{l\to\mathbb{C}}),(\mathrm{r}_i^{r\to n},\mathrm{r}_s^{r\to\mathbb{C}})} = \operatorname{diag}\left(\mathrm{l}_s^{l\to\mathbb{C}} \oplus \imath^{n\to\mathbb{C}} \oplus \mathrm{r}_s^{r\to\mathbb{C}}\right) \mathrm{G}_{\begin{bmatrix} \mathrm{l_i}^{l\to n} \\ \imath_n \\ \mathrm{r_i}^{r\to n} \end{bmatrix}}$$

Definition 59 (Linear Extension)**.**

$$\mathrm{E}^{\mathrm{lin}}_{n,C_l,C_r} = \begin{bmatrix} C_l \\ \mathrm{I}_n \\ C_r \end{bmatrix} \quad , \quad C_l \in \mathbb{C}^{l\times n}, C_r \in \mathbb{C}^{r \times n}$$

2.7.2 Extension Index Mappings and Scaling Functions

Definition 60 (Constant Left).

$$e_l^{\text{const,l}} = \mathop{\left[\,\text{—}\,\right]}\limits_{j=0}^{l-1} (0)_n$$

Definition 61 (Constant Right).

$$e_r^{\text{const,r}} = \mathop{\left[\,\text{—}\,\right]}\limits_{j=0}^{r-1} (n-1)_n$$

Definition 62 (Periodic Left).

$$e_l^{\text{per,l}} = (n-l)_+^{l \to n}$$

Definition 63 (Periodic Right).

$$e_r^{\text{per,r}} = (0)_+^{r \to n}$$

Definition 64 (Half-point Symmetric/Antisymmetric Left).

$$e_l^{\text{hsym,l}} = (0)_+^{l \to n} \circ \jmath_l$$

Definition 65 (Half-point Symmetric/Antisymmetric Right).

$$e_r^{\text{hsym,r}} = (n-r)_+^{r \to n} \circ \jmath_r$$

Definition 66 (Half-point Antisymmetric Left Scaling).

$$s_l^{\text{hasym,l}} = -\imath^{l \to \mathbb{C}}$$

Definition 67 (Half-point Antisymmetric Right Scaling).

$$s_r^{\text{hasym,r}} = -\imath^{r \to \mathbb{C}}$$

Definition 68 (Whole-point Symmetric Left).

$$e_l^{\text{wsym,l}} = (1)_+^{l \to n} \circ \jmath_l$$

Definition 69 (Whole-point Symmetric Right).

$$e_r^{\text{wsym,r}} = (n-r-1)_+^{r \to n} \circ \jmath_r$$

Definition 70 (Whole-point Antisymmetric Left Index Mapping).

$$e_l^{\text{wasym,l}} = (0)_+^{l-1 \to n} \circ \begin{bmatrix} \jmath_{l-1} \\ (0)_{l-1} \end{bmatrix}$$

Definition 71 (Whole-point Antisymmetric Right Index Mapping).

$$e_r^{\text{wasym,r}} = (n-r+1)_+^{r-1 \to n} \circ \begin{bmatrix} \jmath_{r-1} \\ (r-2)_{r-1} \end{bmatrix}$$

Definition 72 (Whole-point Antisymmetric Left Scaling).

$$s_l^{\text{wasym,l}} = -\delta_{\mathbb{I}_{0,l-1}}^{l \to \mathbb{C}}$$

Definition 73 (Whole-point Antisymmetric Right Scaling).

$$s_r^{\text{wasym,r}} = -\delta_{\mathbb{I}_{1,r}}^{r \to \mathbb{C}}$$

3 Function Identities

3.1 Integer Arithmetics

In the following, $a, b, i, j, n \in \mathbb{N}_0$.

Identity 3.1.

$$\lfloor i \rfloor = i$$

Identity 3.2. For $0 \leq i < n$ it holds that

$$\left\lfloor \frac{i}{n} \right\rfloor = 0.$$

Identity 3.3. For $0 \leq i < n$ it holds that

$$\left\lfloor j + \frac{i}{n} \right\rfloor = j.$$

Identity 3.4. For $0 \leq i < n$ and $an \leq b$ it holds that

$$\left\lfloor \frac{j}{a} + \frac{i}{b} \right\rfloor = \left\lfloor \frac{j}{a} \right\rfloor.$$

Identity 3.5. For $a + b < n$ it holds that

$$(a + b) \bmod n = (a \bmod n) + (b \bmod n).$$

Identity 3.6. For $0 \leq i < n$ it holds that

$$(jn + i) \bmod n = i$$

Identity 3.7. For any a and b it holds that

$$(ab) \bmod (an) = a(b \bmod n).$$

Identity 3.8. For any a it holds that

$$\left\lfloor j + \frac{i}{a} \right\rfloor = j + \left\lfloor \frac{i}{a} \right\rfloor.$$

Identity 3.9. For any a it holds that

$$an \bmod n = 0.$$

Identity 3.10. For any a,b, and c it holds that

$$\frac{ai + bj}{c} = i\frac{a}{c} + j\frac{b}{c}$$

Identity 3.11.

$$\lfloor i \bmod k \rfloor = i \bmod k$$

Identity 3.12.

$$\left\lfloor \frac{i}{k} \right\rfloor \bmod k = \left\lfloor \frac{i}{k} \right\rfloor$$

Identity 3.13. The inverse of the function

$$f : \begin{cases} \{0, \ldots, mn - 1\} \to \{0, \ldots, mn - 1\} \\ i \mapsto \lfloor \frac{i}{n} \rfloor + m(i \bmod n) \end{cases}$$

is

$$f^{-1} : \begin{cases} \{0, \ldots, mn - 1\} \to \{0, \ldots, mn - 1\} \\ i \mapsto \lfloor \frac{i}{m} \rfloor + n(i \bmod m) \end{cases}$$

3.2 Sets of Integers

3.2.1 Empty Set

Identity 3.14.

$$M \cup \emptyset = M \quad , \quad M \subset \mathbb{N}_0$$

Identity 3.15.

$$k\emptyset = \emptyset \quad , \quad k \in \mathbb{N}_0$$

Identity 3.16.

$$k + \emptyset = \emptyset \quad , \quad k \in \mathbb{N}_0$$

3.2.2 Intervals

Identity 3.17.

$$\mathbb{I}_{m,n} \cup \mathbb{I}_{n,p} = \mathbb{I}_{m,p}$$

Identity 3.18.

$$(m)_+^{n \to m+n}(\mathbb{I}_n) = \mathbb{I}_{m,m+n}$$

Identity 3.19.

$$(k)_+^{n \to N}(\mathbb{I}_{m,n}) = \mathbb{I}_{m+k,n+k} \quad , \quad n + k < N$$

Identity 3.20.

$$\imath_N(\mathbb{I}_{m,n}) = \mathbb{I}_{m,n} \quad , \quad n \leq N$$

Identity 3.21.

$$\left(\pi^{m\circlearrowleft} \oplus \sigma^{n\circlearrowleft}\right)(\mathbb{I}_m) = \mathbb{I}_m$$

Identity 3.22.

$$\left(\pi^{m\circlearrowleft} \oplus \sigma^{n\circlearrowleft}\right)(\mathbb{I}_{m,m+n}) = \mathbb{I}_{m,m+n}$$

Identity 3.23.

$$(0)_+^{k \to m}(\mathbb{I}_k) = \mathbb{I}_k$$

Identity 3.24.

$$\pi^{n\circlearrowleft}(\mathbb{I}_n) = \mathbb{I}_n$$

Identity 3.25.

$$\mathrm{z}_n^{n-k}(\mathbb{I}_{k,n}) = \mathbb{I}_{0,n-k}$$

Identity 3.26.

$$\left\langle \pi_0^{r\circlearrowleft} \oplus \sigma_0^{n-r\circlearrowleft} | \pi_1^{r\circlearrowleft} \oplus \sigma_1^{n-r\circlearrowleft} \right\rangle_t (\mathbb{I}_r) = \mathbb{I}_r \quad , \quad \forall t \in \mathbb{N}_0$$

Identity 3.27.

$$\left\langle \pi_0^{r\circlearrowleft} \oplus \sigma_0^{n-r\circlearrowleft} | \pi_1^{r\circlearrowleft} \oplus \sigma_1^{n-r\circlearrowleft} \right\rangle_t (\mathbb{I}_{r,n}) = \mathbb{I}_{r,n} \quad , \quad \forall t \in \mathbb{N}_0$$

3.3 Interval Mapping Functions

3.3.1 Function Tensors as Integer Expressions

Property 3.1.

$$(j)_m \otimes \imath_n : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, mn-1\} \\ i \mapsto jn + i \end{cases}$$

Property 3.2. For

$$f : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, N-1\} \\ i \mapsto f(i) \end{cases}$$

it holds that

$$(j)_m \otimes f : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, mN-1\} \\ i \mapsto jN + f(i). \end{cases}$$

Property 3.3.

$$\imath_n \otimes (j)_m : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, mn-1\} \\ i \mapsto j + im \end{cases}$$

Property 3.4. For

$$f : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, N-1\} \\ i \mapsto f(i) \end{cases}$$

it holds that

$$f \otimes (j)_m : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, mN-1\} \\ i \mapsto j + mf(i). \end{cases}$$

Property 3.5.

$$(r)_k \otimes \imath_m \otimes (s)_n : \begin{cases} \{0, \ldots, m-1\} \to \{0, \ldots, kmn-1\} \\ i \mapsto rmn + s + in \end{cases}$$

Property 3.6.

$$\imath_n \otimes (j)_m \otimes \imath_k : \begin{cases} \{0, \ldots, kn-1\} \to \{0, \ldots, kmn-1\} \\ i \mapsto jk + km \left\lfloor \frac{i}{k} \right\rfloor + (i \bmod k) \end{cases}$$

Property 3.7.

$$\left(\imath_n \otimes (j)_m \otimes \imath_k\right) \circ \ell_k^{kn} : \begin{cases} \{0, \ldots, kn-1\} \to \{0, \ldots, kmn-1\} \\ i \mapsto jk + \left\lfloor \frac{i}{n} \right\rfloor + km(i \bmod n) \end{cases}$$

Property 3.8.

$$(j \bmod m)_m \otimes \imath_n \otimes \left(\left\lfloor \tfrac{j}{m} \right\rfloor\right)_k : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, kmn-1\} \\ i \mapsto \left\lfloor \frac{j}{m} \right\rfloor + kn(j \bmod m) + ki \end{cases}$$

with

$$0 \leq j < km$$

Property 3.9.

$$\underbrace{\imath_n \oplus \jmath_n \oplus \cdots}_{m \text{ summands}} : \begin{cases} \{0, \ldots, mn-1\} \to \{0, \ldots, mn-1\} \\ i \mapsto i + \left(\left\lfloor \frac{i}{n} \right\rfloor \bmod 2\right)\left((n-1) - 2(i \bmod n)\right) \end{cases}$$

Property 3.10.

$$\Big(\underbrace{\imath_m \oplus \jmath_m \oplus \cdots}_{m \text{ summands}}\Big) \circ \left(\imath_n \otimes (j)_m\right) : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, mn-1\} \\ i \mapsto j + im + (i \bmod 2)\left((m-1) - 2j\right) \end{cases}$$

Property 3.11. The concatenation of

$$f_j = \Big(\underbrace{\imath_s \oplus \jmath_s \oplus \cdots}_{t \text{ summands}}\Big) \circ \left(\imath_t \otimes (j)_s\right) \quad , \quad 0 \leq j < s$$

and

$$g_k = \big(\underbrace{\imath_r \oplus \jmath_r \oplus \cdots}_{st \text{ summands}}\big) \circ \big(\imath_{st} \otimes (k)_r\big) \quad , \quad 0 \le k < r$$

is given by

$$(g \circ f)_{j,k} : \begin{cases} \{0, \ldots, t-1\} \to \{0, \ldots, rst-1\} \\ i \mapsto k + jr + irs + r\,(i \bmod 2)\big((s-1) - 2j\big) + \big((i+j) \bmod 2\big)\big((r-1) - 2k\big). \end{cases}$$

3.3.2 General Identities

Identity 3.28.

$$\big(a \otimes b\big) \otimes c = a \otimes \big(b \otimes c\big) = a \otimes b \otimes c$$

Identity 3.29.

$$(i)_m \otimes (j)_n = (in + j)_{mn}$$

Identity 3.30.

$$\imath_m \otimes \imath_n = \imath_{mn}$$

Identity 3.31.

$$\imath_m \oplus \imath_n = \imath_{m+n}$$

Identity 3.32.

$$\imath_N \circ f^{n \to N} = f^{n \to N}$$

Identity 3.33.

$$f^{n \to N} \circ \imath_n = f^{n \to N}$$

Identity 3.34.

$$f^{n \to N} \otimes \imath_1 = f^{n \to N}$$

Identity 3.35.

$$\imath_1 \otimes f^{n \to N} = f^{n \to N}$$

Identity 3.36.

$$\left(f_0^{N_0 \to M_0} \otimes \cdots \otimes f_k^{N_k \to M_k}\right) \circ \left(g_0^{n_0 \to N_0} \otimes \cdots \otimes g_k^{n_k \to N_k}\right) = \left(f_0^{N_0 \to M_0} \circ g_0^{n_0 \to N_0}\right) \otimes \cdots \otimes \left(f_k^{N_k \to M_k} \circ g_k^{n_k \to N_k}\right)$$

Identity 3.37.

$$\big(\imath_m \otimes f^{n \to N}\big) \circ \big(\imath_m \otimes g^{n \to N}\big) = \imath_m \otimes \big(f^{n \to N} \circ g^{n \to N}\big)$$

Identity 3.38.

$$f^{m \to M} \otimes g^{n \to N} = \big(f^{m \to M} \otimes \imath_n\big) \circ \big(\imath_m \otimes g^{n \to N}\big)$$

Identity 3.39.

$$f^{m \to M} \otimes g^{n \to N} = \big(\imath_m \otimes g^{n \to N}\big) \circ \big(f^{m \to M} \otimes \imath_n\big)$$

Identity 3.40.

$$a^{n \to N} \circ (j)_n = \big(a(j)\big)_N$$

3.3.3 Pulling in Functions

Identity 3.41.

$$\big(f^{1 \to m} \otimes r^{k \to N} \otimes g^{1 \to n}\big) \circ h^{l \to k} = f^{1 \to m} \otimes (r \circ h)^{l \to N} \otimes g^{1 \to n}$$

Identity 3.42.

$$\big(f^{1 \to m} \otimes h^{n \to N}\big) \circ g^{l \to n} = f^{1 \to m} \otimes (h \circ g)^{l \to N}$$

Identity 3.43.

$$\big(h^{m \to M} \otimes g^{1 \to n}\big) \circ f^{k \to m} = (h \circ f)^{k \to M} \otimes g^{1 \to n}$$

Pulling in Functions–Special Cases

Identity 3.44.

$$\big(\imath_{km} \otimes (j)_n\big) \circ \big((i)_k \otimes \imath_m\big) = (i)_k \otimes \imath_m \otimes (j)_n$$

Identity 3.45.

$$\big((i)_k \otimes \imath_{mn}\big) \circ \big(\imath_m \otimes (j)_n\big) = (i)_k \otimes \imath_m \otimes (j)_n$$

Identity 3.46.

$$\big(\imath_m \otimes (j)_n\big) \circ f^{k\to m} = f^{k\to m} \otimes (j)_n.$$

Identity 3.47.

$$\big((j)_n \otimes \imath_m\big) \circ f^{k\to m} = (j)_n \otimes f^{k\to m}.$$

Identity 3.48.

$$\big(f^{1\to m} \otimes \imath_k \otimes g^{1\to n}\big) \circ h^{l\to k} = f^{1\to m} \otimes h^{l\to k} \otimes g^{1\to n}$$

Identity 3.49.

$$\big(f^{1\to m} \otimes \imath_n\big) \circ g^{l\to n} = f^{1\to m} \otimes g^{l\to n}$$

Identity 3.50.

$$\big(\imath_m \otimes g^{1\to n}\big) \circ f^{k\to m} = f^{k\to m} \otimes g^{1\to n}$$

3.3.4 Stride Permutation Identities

Identity 3.51.

$$\ell_m^{mn} \circ \big((j)_m \otimes f^{k\to n}\big) = f^{k\to n} \otimes (j)_m$$

Identity 3.52.

$$\ell_n^{mn} \circ \big(f^{k\to n} \otimes (j)_m\big) = (j)_m \otimes f^{k\to n}$$

Stride Permutation Identities–Special Cases

Identity 3.53.

$$\ell_m^{mn} \circ \big((j)_m \otimes \imath_n\big) = \imath_n \otimes (j)_m$$

Identity 3.54.

$$\ell_n^{mn} \circ \big(\imath_n \otimes (j)_m\big) = (j)_m \otimes \imath_n$$

Stride Permutation Stacking Identities

Identity 3.55.

$$\ell_k^{kmn} \circ \big((j)_{km} \otimes \imath_n\big) = (j \bmod m)_m \otimes \imath_n \otimes \big(\lfloor \tfrac{j}{m} \rfloor\big)_k$$

Identity 3.56 (Stacking/Splitting)**.**

$$\mathop{\left[\!\!\text{---}\!\!\right]}_{i=jk}^{(j+1)k-1} \imath_n \otimes (i)_{km} = \big(\imath_n \otimes (j)_m \otimes \imath_k\big) \circ \ell_k^{kn}$$

Identity 3.57 (Stacking/Splitting)**.**

$$\mathop{\left[\!\!\text{---}\!\!\right]}_{i=jk}^{(j+1)k-1} (i)_{km} \otimes \imath_n = (j)_m \otimes \imath_{kn}$$

Identity 3.58 (Stacking/Splitting)**.**

$$\mathop{\left[\!\!\text{---}\!\!\right]}_{i=jk}^{(j+1)k-1} (i \bmod k)_k \otimes \imath_n \otimes \big(\lfloor \tfrac{i}{k} \rfloor\big)_m = \imath_{kn} \otimes (j)_m$$

3.3.5 K/M Permutation Identities

Identity 3.59.

$$\big(\underbrace{\imath_m \oplus \jmath_m \oplus \cdots}_{n \text{ summands}}\big) = \underbrace{\imath_n \otimes \langle \imath_m | \jmath_m \rangle_{\lfloor \frac{(.)}{m} \rfloor}}_{.}$$

Identity 3.60.

$$\big(\underbrace{\imath_n \otimes \pi^{m\circlearrowleft}_{\lfloor \frac{(.)}{m}\rfloor}}_{.}\big) \circ \big(\imath_n \otimes g^{1\to m}\big) = \underbrace{\imath_n \otimes \big(\pi^{m\circlearrowleft}_{(.)} \circ g^{1\to m}\big)}_{.}$$

Identity 3.61.

$$\underbrace{\imath_n \otimes \big(\langle \imath_m|\jmath_m\rangle_{(.)}(j)\big)_m}_{.} = \underbrace{\imath_n \otimes \mathrm{x}^{j,(.)}_m}_{.}$$

Identity 3.62. For

$$f = \underbrace{\imath_{tu_0 \ldots u_q} \otimes \mathrm{x}^{j_0,m_0+(.)}_{s_0} \otimes \ldots \otimes \mathrm{x}^{j_r,m_r+(.)}_{s_r}}_{.}$$

and

$$g = \underbrace{\imath_t \otimes \mathrm{x}^{k_0,n_0+(.)}_{u_0} \otimes \ldots \otimes \mathrm{x}^{k_q,n_q+(.)}_{u_q}}_{.}$$

it holds that

$$f \circ g = \underbrace{\imath_t \otimes \mathrm{x}^{k_0,n_0+(.)}_{u_0} \otimes \ldots \otimes \mathrm{x}^{k_q,n_q+(.)}_{u_q} \otimes \mathrm{x}^{j_0,n_q+k_q+m_0+(.)}_{s_0} \otimes \ldots \otimes \mathrm{x}^{j_r,n_q+k_q+m_r+(.)}_{s_r}}_{.}$$

K/K Permutation Derived Identities

Identity 3.63.

$$\mathrm{m}^{mn}_m \circ \big((j)_m \otimes \imath_n\big) = \big(\underbrace{\imath_m \oplus \jmath_m \oplus \cdots}_{n \text{ summands}}\big) \circ \big(\imath_n \otimes (j)_m\big)$$

Identity 3.64.

$$\mathrm{m}^{mn}_m \circ \big((j)_m \otimes f^{k\to n}\big) = \Big(\big(\underbrace{\imath_m \oplus \jmath_m \oplus \cdots}_{n \text{ summands}}\big) \circ \big(\imath_n \otimes (j)_m\big)\Big) \circ f^{k\to n}$$

Identity 3.65.

$$\big(\underbrace{\imath_m \oplus \jmath_m \oplus \cdots}_{n \text{ summands}}\big) \circ \big(\imath_n \otimes (j)_m\big) = \underbrace{\imath_n \otimes \mathrm{x}^{j,(.)}_m}_{.}$$

Identity 3.66.

$$\Big(\underbrace{\imath_{st} \otimes \mathrm{x}^{k,(.)}_r}_{.}\Big) \circ \Big(\underbrace{\imath_t \otimes \mathrm{x}^{j,(.)}_s}_{.}\Big) = \underbrace{\imath_t \otimes \mathrm{x}^{j,(.)}_s \otimes \mathrm{x}^{k,j+(.)}_r}_{.}$$

Identity 3.67.

$$\Big(\underbrace{\imath_{stu} \otimes \mathrm{x}^{k,(.)}_r}_{.}\Big) \circ \Big(\underbrace{f^{t\to tu} \otimes \mathrm{x}^{j,m+(.)}_s}_{.}\Big) = \underbrace{f^{t\to tu} \otimes \mathrm{x}^{j,m+(.)}_s \otimes \mathrm{x}^{k,j+m+(.)}_r}_{.}$$

3.3.6 Gamma Product

Identity 3.68.

$$\left(f^{m\to M}\boxtimes g^{n\to N}\right)\circ\left(r^{m'\to m}\otimes s^{n'\to n}\right)=\left(f^{m\to M}\circ r^{m'\to m}\right)\boxtimes\left(g^{n\to N}\circ s^{n'\to n}\right)$$

Identity 3.69.

$$\imath_r\boxtimes(j)_s=\mathrm{z}_{rs}^{(r-1)j}\circ\big(\imath_r\otimes(j)_s\big)$$

Identity 3.70.

$$(j)_r\boxtimes\imath_s=\mathrm{z}_{rs}^{(s-1)j}\circ\big(\imath_s\otimes(j)_r\big)$$

Identity 3.71.

$$\mathrm{z}_{mn}^{k}\circ\big(f^{m'\to m}\otimes(j)_n\big)=\left(\mathrm{z}_m^{\left\lfloor\frac{k+j}{n}\right\rfloor}\circ f^{m'\to m}\right)\otimes\left(\mathrm{z}_n^k\circ(j)_n\right)$$

Identity 3.72.

$$\bar{\ell}_c^{mn}\circ\left(\bar{\ell}_a^m\boxtimes\bar{\ell}_b^n\right)=\bar{\ell}_{ac}^m\boxtimes\bar{\ell}_{bc}^n$$

Identity 3.73.

$$\bar{\ell}_k^{mn}\circ\left(f^{m'\to m}\boxtimes g^{n'\to n}\right)=\left(\bar{\ell}_k^m\circ f\right)\boxtimes\left(\bar{\ell}_k^n\circ g\right)$$

Identity 3.74.

$$\bar{\ell}_r^n\circ\bar{\ell}_s^n=\bar{\ell}_{rs}^n$$

Identity 3.75.

$$\bar{\ell}_r^N\circ w_{\varphi,g}^{n\to N}=w_{r\varphi,g}^{n\to N}$$

Identity 3.76.

$$\bar{\ell}_r^n\circ\mathrm{z}_n^k=\mathrm{z}_n^{kr}\circ\bar{\ell}_r^n$$

Identity 3.77.

$$\mathrm{z}_n^k\circ\bar{\ell}_r^n=\bar{\ell}_r^n\circ\mathrm{z}_n^{kr^{-1}}$$

3.4 Permutation Generating Functions

3.4.1 Inverting Permutations

Identity 3.78.

$$\imath_n=\left(\imath_n\right)^{-1}$$

Identity 3.79.

$$\jmath_n=\left(\jmath_n\right)^{-1}$$

Identity 3.80.

$$\ell_m^{mn}=\left(\ell_n^{mn}\right)^{-1}$$

Identity 3.81.

$$\mathrm{k}_m^{mn}=\left(\mathrm{m}_n^{mn}\right)^{-1}$$

Identity 3.82.

$$\mathrm{m}_m^{mn}=\left(\mathrm{k}_n^{mn}\right)^{-1}$$

Identity 3.83. For two permutation generating functions

$$\pi\in\mathrm{S}_m\quad\text{and}\quad w\in\mathrm{S}_n$$

it holds that

$$(\pi\otimes w)^{-1}=\pi^{-1}\otimes w^{-1}.$$

Identity 3.84. For two permutation generating functions

$$\pi \in \mathrm{S}_m \quad \text{and} \quad w \in \mathrm{S}_n$$

it holds that

$$(\pi \circ w)^{-1} = w^{-1} \circ \pi^{-1}.$$

Identity 3.85. For two permutation generating functions

$$\pi \in \mathrm{S}_m \quad \text{and} \quad w \in \mathrm{S}_n$$

it holds that

$$(\pi \oplus w)^{-1} = \pi^{-1} \oplus w^{-1}.$$

3.4.2 Dot and Alternator Identities for Permutations

Identity 3.86.

$$\langle \pi | \pi \rangle_t = \pi$$

Identity 3.87.

$$\langle \pi | \sigma \rangle_t^{-1} = \langle \pi^{-1} | \sigma^{-1} \rangle_t$$

Identity 3.88.

$$\langle \pi_0 | \sigma_0 \rangle_t \circ \langle \pi_1 | \sigma_1 \rangle_t = \langle \pi_0 \circ \pi_1 | \sigma_0 \circ \sigma_1 \rangle_t$$

Identity 3.89.

$$\langle \pi_0 | \sigma_0 \rangle_t \oplus \langle \pi_1 | \sigma_1 \rangle_t = \langle \pi_0 \oplus \pi_1 | \sigma_0 \oplus \sigma_1 \rangle_t$$

Identity 3.90.

$$\langle \pi_0 | \sigma_0 \rangle_t \otimes \langle \pi_1 | \sigma_1 \rangle_t = \langle \pi_0 \otimes \pi_1 | \sigma_0 \otimes \sigma_1 \rangle_t$$

Identity 3.91.

$$\langle \pi | \sigma \rangle_s \circ \langle \tau | w \rangle_t = \Big\langle \, \langle \pi \circ \tau | \sigma \circ \tau \rangle_s \, \big| \, \langle \pi \circ w | \sigma \circ w \rangle_s \, \Big\rangle_t$$

Identity 3.92.

$$\langle \pi | \sigma \rangle_s \circ \langle \tau | w \rangle_t = \Big\langle \, \langle \pi \circ \tau | \pi \circ w \rangle_t \, \big| \, \langle \sigma \circ \tau | \sigma \circ w \rangle_t \, \Big\rangle_s$$

Identity 3.93.

$$\langle \sigma | \tau \rangle_t \circ \langle w | \kappa \rangle_{\pi^{n\circlearrowleft}(t)} = \langle \sigma \circ w | \tau \circ \kappa \rangle_t \quad \text{iff} \quad \pi(i) \equiv i \bmod 2 \quad \forall i \in \mathbb{I}_n$$

Identity 3.94.

$$\langle \sigma | \tau \rangle_t \circ \langle w | \kappa \rangle_{\pi^{n\circlearrowleft}(t)} = \langle \sigma \circ \kappa | \tau \circ w \rangle_t \quad \text{iff} \quad \pi(i) \equiv i + 1 \bmod 2 \quad \forall i \in \mathbb{I}_n$$

Identity 3.95.

$$\langle \sigma | \tau \rangle_t \circ \langle w | \kappa \rangle_{\pi^{n\circlearrowleft}(t)} = \langle \tau \circ w | \sigma \circ \kappa \rangle_{\pi^{n\circlearrowleft}(t)} \quad \text{iff} \quad \pi(i) \equiv i + 1 \bmod 2 \quad \forall i \in \mathbb{I}_n$$

Property 3.12. For

$$\pi^{n\circlearrowleft} \in \left\{ z_n^{2k}, \imath_n, \jmath_{2k+1} \right\}$$

it holds that

$$\pi(i) \equiv i \bmod 2 \quad \forall i \in \mathbb{I}_n$$

Property 3.13. For

$$\pi^{n\circlearrowleft} \in \left\{ z_n^{2k+1}, \jmath_{2k} \right\}$$

it holds that

$$\pi(i) \equiv i + 1 \bmod 2 \quad \forall i \in \mathbb{I}_n$$

Identity 3.96.

$$\left(\pi^{m\circlearrowleft} \otimes \imath_n\right) \circ \underbrace{\left(\imath_m \otimes \sigma^{n\circlearrowleft}_{\left\lfloor \frac{(.)}{n} \right\rfloor}\right)}_{.} = \underbrace{\pi^{m\circlearrowleft} \otimes \sigma^{n\circlearrowleft}_{\pi\left(\left\lfloor \frac{(.)}{n} \right\rfloor\right)}}_{.}$$

Identity 3.97.

$$\underbrace{\left(\imath_m \otimes \tau^{n\circlearrowleft}_{\left\lfloor \frac{(.)}{n} \right\rfloor}\right)}_{.} \circ \left(\pi^{m\circlearrowleft} \otimes \imath_n\right) = \underbrace{\pi^{m\circlearrowleft} \otimes \tau^{n\circlearrowleft}_{\left\lfloor \frac{(.)}{n} \right\rfloor}}_{.}$$

Identity 3.98.

$$\underbrace{\left(\imath_m \otimes \tau^{n\circlearrowleft}_{\left\lfloor \frac{(.)}{n} \right\rfloor}\right)}_{.} \circ \underbrace{\left(\pi^{m\circlearrowleft} \otimes \sigma^{n\circlearrowleft}_{\pi\left(\left\lfloor \frac{(.)}{n} \right\rfloor\right)}\right)}_{.} = \underbrace{\pi^{m\circlearrowleft} \otimes \left(\tau^{n\circlearrowleft}_{\left\lfloor \frac{(.)}{n} \right\rfloor} \circ \sigma^{n\circlearrowleft}_{\pi\left(\left\lfloor \frac{(.)}{n} \right\rfloor\right)}\right)}_{.}$$

Identity 3.99.

$$\underbrace{\left(\pi^{m\circlearrowleft} \otimes \tau^{n\circlearrowleft}_{\left\lfloor \frac{(.)}{n} \right\rfloor}\right)}_{.} \circ \underbrace{\left(\imath_m \otimes \sigma^{n\circlearrowleft}_{\left\lfloor \frac{(.)}{n} \right\rfloor}\right)}_{.} = \underbrace{\pi^{m\circlearrowleft} \otimes \left(\tau^{n\circlearrowleft}_{\left\lfloor \frac{(.)}{n} \right\rfloor} \circ \sigma^{n\circlearrowleft}_{\pi\left(\left\lfloor \frac{(.)}{n} \right\rfloor\right)}\right)}_{.}$$

Identity 3.100.

$$\left(\pi^{m\circlearrowleft} \otimes \imath_n\right)^{\overbrace{\imath_m \otimes \tau^{n\circlearrowleft}_{\left\lfloor \frac{(.)}{n} \right\rfloor}}^{.}} = \underbrace{\pi^{m\circlearrowleft} \otimes \left(\tau^{n\circlearrowleft}_{\left\lfloor \frac{(.)}{n} \right\rfloor} \circ \left(\tau^{-1}\right)^{n\circlearrowleft}_{\pi\left(\left\lfloor \frac{(.)}{n} \right\rfloor\right)}\right)}_{.}$$

3.4.3 Stride Permutation

Identity 3.101.

$$\ell^n_r \circ \ell^n_s = \ell^n_{rs}$$

Identity 3.102.

$$\ell^m_m = \ell^m_1$$

Identity 3.103.

$$\ell^m_1 = \imath_m$$

Identity 3.104.

$$\ell^{mn}_m \circ \jmath_{mn} = \ell^{mn}_m$$

Identity 3.105.

$$\jmath_{mn} \circ \ell^{mn}_m = \ell^{mn}_m$$

Identity 3.106.

$$\left(\pi^{m\circlearrowleft} \otimes \sigma^{n\circlearrowleft}\right)^{\ell^{mn}_m} = \sigma^{n\circlearrowleft} \otimes \pi^{m\circlearrowleft}$$

3.4.4 Cyclic Shift Identities

Identity 3.107.

$$\left(\mathrm{z}^k_n\right)^{-1} = \mathrm{z}^{m-k}_n$$

Identity 3.108.

$$\mathrm{z}^m_m = \mathrm{z}^0_m$$

Identity 3.109.

$$\mathrm{z}^0_m = \imath_m$$

Identity 3.110.

$$\mathrm{z}^k_n \; \mathrm{z}^m_n = \mathrm{z}^{k+m}_n$$

3.4.5 Derived Cyclic Shift Identities

Identity 3.111.

$$\left(\mathrm{z}_m^{2k} \otimes \imath_n\right)^{\overbrace{\imath_n \oplus \pi^{n\circlearrowleft} \oplus \cdots}^{}}_{\text{m summands}} = \mathrm{z}_m^{2k} \otimes \imath_n$$

Identity 3.112.

$$\left(\mathrm{z}_m^{2k+1} \otimes \imath_n\right)^{\overbrace{\imath_n \oplus \pi^{n\circlearrowleft} \oplus \cdots}^{}}_{\text{m summands}} = \mathrm{z}_m^{2k+1} \otimes \pi^{n\to n}$$

3.4.6 Permutations and Add Function

Identity 3.113.

$$\left(\pi^{m\circlearrowleft} \oplus \sigma^{n\circlearrowleft}\right) \circ (m)_+^{n\to m+n} = \circ(m)_+^{n\to m+n} \circ \sigma^{n\circlearrowleft}$$

Identity 3.114.

$$\mathrm{z}_m^k \circ (m-k)_+^{k\to m} = (0)_+^{k\to m}$$

3.5 Matrix Generating Functions

3.5.1 Constant Functions

Identity 3.115.

$$\imath^{n\to\mathbb{C}} = (1)^{n\to\mathbb{C}}$$

Identity 3.116.

$$\mathrm{o}^{n\to\mathbb{C}} = (0)^{n\to\mathbb{C}}$$

Identity 3.117.

$$(c)^{n\to\mathbb{C}} = c\imath^{n\to\mathbb{C}}$$

Identity 3.118.

$$(c)^{N\to\mathbb{C}} \circ f^{n\to N} = (c)^{n\to\mathbb{C}} \quad , \quad c \in \mathbb{C}$$

Identity 3.119.

$$(c)^{m\to\mathbb{C}} \otimes (c)^{n\to\mathbb{C}} = (c^2)^{mn\to\mathbb{C}} \quad , \quad c \in \mathbb{C}$$

Identity 3.120.

$$(c)^{m\to\mathbb{C}} \oplus (c)^{n\to\mathbb{C}} = (c)^{m+n\to\mathbb{C}} \quad , \quad c \in \mathbb{C}$$

Identity 3.121.

$$c\imath^{m\to\mathbb{C}} \oplus d\imath^{n\to\mathbb{C}} = c\delta_{\mathbb{I}_m}^{m+n} + d\delta_{\mathbb{I}_{m,n}}^{m+n} \quad , \quad c, d \in \mathbb{C}$$

3.5.2 Delta Function

Identity 3.122.

$$\delta_\emptyset^n = \mathrm{o}^{n\to\mathbb{C}}$$

Identity 3.123.

$$\delta_{\mathbb{I}_m}^n = \imath^{n\to\mathbb{C}}$$

Identity 3.124.

$$\delta_M^m \oplus \delta_N^n = \delta_{M\cup(m+N)}^{m+n}$$

Identity 3.125.

$$\delta_M^m \, \delta_{M'}^m = \delta_{M\cap M'}^m$$

Identity 3.126.

$$\delta_I^N \circ f^{n\to N} = \delta_{f^{-1}(I\cap f(\mathbb{I}_n))}^n$$

Identity 3.127.

$$\delta_N^n \circ \pi^{n\circlearrowleft} = \delta^n_{(\pi^{n\circlearrowleft})^{-1}(N)}$$

Identity 3.128.

$$\delta_M^m \oplus \mathrm{o}^{n\to\mathbb{C}} = \delta_M^{m+n}$$

Identity 3.129.

$$\mathrm{o}^{n\to\mathbb{C}} \oplus \delta_M^m = \delta^{m+n}_{(n)_+^{m\to m+n}(M)}$$

Identity 3.130.

$$\delta_M^m \otimes \delta_N^n = \delta^{mn}_{((M)_m \otimes \imath_n)(N)}$$

Identity 3.131.

$$\delta_M^m \otimes \delta_N^n = \delta^{mn}_{(\imath_m \otimes (N)_n)(M)}$$

Identity 3.132.

$$\left(\delta_M^m \otimes \delta_N^n\right) \circ \underbrace{\left(\imath_m \otimes \pi^{n\circlearrowleft}_{\left\lfloor \frac{(.)}{n} \right\rfloor} \right)}_{.} = \underbrace{\delta_M^m \otimes \left(\delta_N^n \circ \pi^{n\circlearrowleft}_{\left\lfloor \frac{(.)}{n} \right\rfloor}\right)}_{.}$$

Delta Function–Derived Identities

Identity 3.133.

$$\delta^n_{\mathbb{I}_r} \circ \left\langle \pi_0^{r\circlearrowleft} \oplus \sigma_0^{n-r\circlearrowleft} | \pi_1^{r\circlearrowleft} \oplus \sigma_1^{n-r\circlearrowleft} \right\rangle_t = \delta^n_{\mathbb{I}_r} \quad , \quad \forall t \in \mathbb{N}_0$$

Identity 3.134.

$$\delta^n_{\mathbb{I}_{r,n}} \circ \left\langle \pi_0^{r\circlearrowleft} \oplus \sigma_0^{n-r\circlearrowleft} | \pi_1^{r\circlearrowleft} \oplus \sigma_1^{n-r\circlearrowleft} \right\rangle_t = \delta^n_{\mathbb{I}_{r,n}} \quad , \quad \forall t \in \mathbb{N}_0$$

Identity 3.135.

$$\left(\delta_M^m \otimes \delta^n_{\mathbb{I}_r}\right) \circ \left(\imath_n \oplus (\pi^{r\circlearrowleft} \oplus \sigma^{n-r\circlearrowleft}) \oplus \cdots\right) = \delta_M^m \otimes \delta^n_{\mathbb{I}_r}$$

Identity 3.136.

$$\left(\delta_M^m \otimes \delta^n_{\mathbb{I}_{r,n}}\right) \circ \left(\imath_n \oplus (\pi^{r\circlearrowleft} \oplus \sigma^{n-r\circlearrowleft}) \oplus \cdots\right) = \delta_M^m \otimes \delta^n_{\mathbb{I}_{r,n}}$$

Identity 3.137.

$$\imath^{m\to\mathbb{C}} \otimes \delta_N^n = \delta^{mn}_{((\mathbb{I}_m)_m \otimes \imath_n)(N)}$$

Identity 3.138.

$$\delta_M^m \otimes \imath^{n\to\mathbb{C}} = \delta^{mn}_{(\imath_m \otimes (\mathbb{I}_n)_n)(M)}$$

Identity 3.139.

$$\delta^m_{\mathbb{I}_{k,n}} \circ \mathrm{z}^r_m = \delta^m_{\mathbb{I}_{k-r,n-r}} \quad , \quad k \geq r$$

Identity 3.140.

$$\delta^m_{\mathbb{I}_{k,n}} \circ \mathrm{z}^r_m = \delta^m_{\mathbb{I}_{k+m-r,n+m-r}} \quad , \quad r \geq n$$

Identity 3.141.

$$\delta^n_{\mathbb{I}_{n-k}} \circ \left(\pi^{n-k\circlearrowleft} \oplus \sigma^{k\circlearrowleft}\right) = \delta^n_{\mathbb{I}_{n-k}}$$

Identity 3.142.

$$\delta^n_{\mathbb{I}_{k,n}} \circ \left(\pi^{k\circlearrowleft} \oplus \sigma^{n-k\circlearrowleft}\right) = \delta^n_{\mathbb{I}_{k,n}}$$

3.5.3 Twiddle Function

Identity 3.143.

$$\mathrm{t}_n^{mn} = \mathrm{t}_n^{kmn} \circ \left((0)_k \otimes \imath_{mn}\right)$$

Identity 3.144.

$$\mathrm{t}_n^{mn} = \mathrm{t}_m^{mn} \circ \ell_n^{mn}$$

4 Matrix Identities

4.1 Formula Constructs and Sum Notation

Identity 4.1 (Direct Sum of Rotations).

$$\bigoplus_{j=0}^{k-1} \mathrm{R}_{\alpha_i} = \overline{\operatorname{diag}\left(k \to \mathbb{C} : i \to e^{i\alpha_i}\right)}$$

Identity 4.2 (Iterative Direct Sum).

$$\bigoplus_{j=0}^{k-1} A_j = \sum_{j=0}^{k-1} \mathrm{S}_{j\otimes\imath_m}\, A_j\, \mathrm{G}_{j\otimes\imath_n} \quad \text{with} \quad A_j \in \mathbb{C}^{m\times n}$$

Identity 4.3 (Iterative Row Overlapped Direct Sum).

$$\bigoplus_{j=0}^{k-1}{}_r A_j = \sum_{j=0}^{k-1} \mathrm{S}^{mk,m}_{j\otimes\imath_m}\, A_j\, \mathrm{G}^{(n-r)k+r,n}_{j\otimes_r\imath_n} \quad \text{with} \quad A_j \in \mathbb{C}^{m\times n}$$

Identity 4.4 (Iterative Column Overlapped Direct Sum).

$$\bigoplus_{j=0}^{k-1}{}^r A_j = \sum_{j=0}^{k-1} \mathrm{S}^{(m-r)k+r,m}_{j\otimes_r\imath_m}\, A_j\, \mathrm{G}^{nk,n}_{j\otimes\imath_n} \quad \text{with} \quad A_j \in \mathbb{C}^{m\times n}$$

Identity 4.5 (Parallel Tensor Product).

$$\mathrm{I}_k \otimes A = \sum_{j=0}^{k-1} \mathrm{S}_{j\otimes\imath_m}\, A\, \mathrm{G}_{j\otimes\imath_n} \quad \text{with} \quad A \in \mathbb{C}^{m\times n}$$

Identity 4.6 (Row Overlapped Tensor Product).

$$\mathrm{I}_k \otimes_r A = \sum_{j=0}^{k-1} \mathrm{S}_{j\otimes\imath_m}\, A\, \mathrm{G}_{((n-r)j)_+^{n\to(n-r)k+r}} \quad \text{with} \quad A \in \mathbb{C}^{m\times n}$$

Identity 4.7 (Column Overlapped Tensor Product).

$$\mathrm{I}_k \otimes^r A = \sum_{j=0}^{k-1} \mathrm{S}_{((m-r)j)_+^{m\to(m-r)k+r}}\, A\, \mathrm{G}_{j\otimes\imath_n} \quad \text{with} \quad A \in \mathbb{C}^{m\times n}$$

Identity 4.8 (Vector Tensor Product).

$$A \otimes \mathrm{I}_k = \sum_{j=0}^{k-1} \mathrm{S}_{\imath_m\otimes j}\, A\, \mathrm{G}_{\imath_n\otimes j} \quad \text{with} \quad A \in \mathbb{C}^{m\times n}$$

Identity 4.9 (Horizontal Stack of Matrices).

$$\Big[\;\Big|\;\Big]_{j=0}^{S-1} A_j = \sum_{j=0}^{S-1} \mathrm{S}_{\imath_m}\, A_j\, \mathrm{G}_{j\otimes\imath_n} \quad \text{with} \quad A_j \in \mathbb{C}^{m\times n}$$

Identity 4.10 (Vertical Stack of Matrices).

$$\Big[\,\text{—}\,\Big]_{j=0}^{R-1} A_j = \sum_{j=0}^{R-1} \mathrm{S}_{j\otimes\imath_m}\, A_j\, \mathrm{G}_{\imath_n} \quad \text{with} \quad A_j \in \mathbb{C}^{m\times n}$$

Identity 4.11 (Matrix of Matrices).

$$\begin{bmatrix} A_{0,0} & \cdots & A_{0,S-1} \\ \vdots & \ddots & \vdots \\ A_{R-1,0} & \cdots & A_{R-1,S-1} \end{bmatrix} = \sum_{j=0}^{R-1}\sum_{k=0}^{S-1} \mathrm{S}_{j\otimes\imath_m}\, A_{j,k}\, \mathrm{G}_{k\otimes\imath_n} \quad \text{with} \quad A_{j,k} \in \mathbb{C}^{m\times n}$$

Identity 4.12 (Product of Scatter and Gather).

$$\mathrm{S}_{w^{n\to N}}\, \mathrm{G}_{r^{n\to N}} = \sum_{j=0}^{n-1} \mathrm{S}_{w^{n\to N}\circ(j)_n}\, \mathrm{G}_{r^{n\to N}\circ(j)_n}$$

Identity 4.13 (Product of Scatter and Gather).

$$\mathrm{S}_{w^{n\to N}} \left(\sum_{k=0}^{m-1} \mathrm{G}_{r_k^{n\to N}} \right) = \sum_{j=0}^{n-1} \left(\mathrm{S}_{w^{n\to N}\circ(j)_n} \sum_{k=0}^{m-1} \mathrm{G}_{r_k^{n\to N}\circ(j)_n} \right)$$

4.2 Gather and Scatter Identities

4.2.1 Gather and Scatter

Identity 4.14 (Trivial Gather Matrix).

$$\mathrm{G}_{\imath_n} = \mathrm{I}_n$$

Identity 4.15 (Trivial Scatter Matrix).

$$\mathrm{S}_{\imath_n} = \mathrm{I}_n$$

Identity 4.16 (Gather Transposition).

$$\left(\mathrm{G}_{f^{n\to N}}\right)^\top = \mathrm{S}_{f^{n\to N}}$$

Identity 4.17 (Scatter Transposition).

$$\left(\mathrm{S}_{f^{n\to N}}\right)^\top = \mathrm{G}_{f^{n\to N}}$$

Identity 4.18 (Gather/Scatter Identity).

$$\mathrm{G}_{f^{n\to N}}\ \mathrm{S}_{f^{n\to N}} = \mathrm{I}_n$$

Identity 4.19 (Scatter/Gather Identity).

$$\mathrm{S}_{f^{n\to N}}\ \mathrm{G}_{f^{n\to N}} = \mathrm{diag}\left(\delta_{f^{n\to N}}\right)$$

Identity 4.20 (Gather Multiplicativity).

$$\mathrm{G}_{s^{n\to N_1}}\ \mathrm{G}_{r^{N_1\to N}} = \mathrm{G}_{(r\circ s)^{n\to N}}$$

Identity 4.21 (Scatter Multiplicativity).

$$\mathrm{S}_{v^{N_1\to N}}\ \mathrm{S}_{w^{n\to N_1}} = \mathrm{S}_{(v\circ w)^{n\to N}}$$

Identity 4.22 (Gather/Scatter Multiplicativity).

$$\mathrm{G}_{r^{n\to N}}\ \mathrm{S}_{w^{n\to N}} = \mathrm{perm}\left(\left(w^{-1}\circ r\right)^{n\to n}\right) \quad \text{for} \quad r\big(\{0,\ldots,n-1\}\big) = w\big(\{0,\ldots,n-1\}\big)$$

Identity 4.23 (Gather/Scatter Multiplicativity)**.** For

$$r : \begin{cases} \{0, \ldots, m\} \to \{0, \ldots, M-1\} \\ i \mapsto r(i), \end{cases}$$

r injective, $0 \le j < M$, and $0 \le k < m$ it holds that

$$\mathrm{G}_{r^{m \to M} \otimes \imath_n} \, \mathrm{S}_{(j)_M \otimes \imath_n} = \begin{cases} \mathrm{S}_{(k)_m \otimes \imath_n} & \text{if } j = r(k) \\ 0^{mn \times n} & \text{else} \end{cases}$$

Identity 4.24 (Gather/Scatter Multiplicativity)**.** For

$$w : \begin{cases} \{0, \ldots, m\} \to \{0, \ldots, M-1\} \\ i \mapsto w(i), \end{cases}$$

w injective, $0 \le j < M$, and $0 \le k < m$ it holds that

$$\mathrm{G}_{(j)_M \otimes \imath_n} \, \mathrm{S}_{w^{m \to M} \otimes \imath_n} = \begin{cases} \mathrm{G}_{(k)_m \otimes \imath_n} & \text{if } j = w(k) \\ 0^{n \times mn} & \text{else} \end{cases}$$

Identity 4.25 (Gather Tensor)**.**

$$\mathrm{G}_{r^{m \to M}} \otimes \mathrm{G}_{s^{n \to N}} = \mathrm{G}_{(r \otimes s)^{mn \to MN}}$$

Identity 4.26 (Scatter Tensor)**.**

$$\mathrm{S}_{r^{m \to M}} \otimes \mathrm{S}_{s^{n \to N}} = \mathrm{S}_{(r \otimes s)^{mn \to MN}}$$

Identity 4.27 (Gather/Scatter Tensor)**.**

$$\left(\mathrm{S}_{w^{m \to M}} \, \mathrm{G}_{r^{m \to M}} \right) \otimes \mathrm{G}_{s^{n \to N}} = \mathrm{S}_{(w \otimes \imath_n)^{mn \to Mn}} \, \mathrm{G}_{(r \otimes s)^{mn \to MN}}$$

Identity 4.28 (Gather Stacking)**.**

$$\left[\frac{\mathrm{G}_{r^{n_1 \to N}}}{\mathrm{G}_{s^{n_2 \to N}}} \right] = \mathrm{G}_{\left[\begin{smallmatrix} r \\ s \end{smallmatrix} \right]^{n_1 + n_2 \to N}}$$

Identity 4.29 (Scatter Stacking)**.**

$$\left[\; \mathrm{S}_{v^{n_1 \to N}} \; \middle| \; \mathrm{S}_{w^{n_2 \to N}} \; \right] = \mathrm{S}_{\left[\begin{smallmatrix} v \\ w \end{smallmatrix} \right]^{n_1 + n_2 \to N}}$$

4.2.2 Gather/Scatter and Permutations

Identity 4.30 (Permutation as Gather Matrix)**.**

$$\mathrm{G}_{\pi^{N \circlearrowleft}} = \mathrm{perm}\left(\pi^{N \circlearrowleft} \right)$$

Identity 4.31 (Permutation as Scatter Matrix)**.**

$$\mathrm{S}_{\pi^{N \circlearrowleft}} = \mathrm{perm}\left((\pi^{-1})^{N \circlearrowleft} \right)$$

Identity 4.32 (Gather/Permutation Multiplicativity)**.**

$$\mathrm{G}_{r^{n \to N}} \; \mathrm{perm}\left(\pi^{N \circlearrowleft} \right) = \mathrm{G}_{(\pi \circ r)^{n \to N}}$$

Identity 4.33 (Permutation/Gather Multiplicativity)**.**

$$\mathrm{perm}\left(\pi^{n \circlearrowleft} \right) \; \mathrm{G}_{r^{n \to N}} = \mathrm{G}_{(r \circ \pi)^{n \to N}}$$

Identity 4.34 (Scatter/Permutation Multiplicativity)**.**

$$\mathrm{S}_{w^{n \to N}} \; \mathrm{perm}\left(\pi^{n \circlearrowleft} \right) = \mathrm{S}_{(w \circ \pi^{-1})^{n \to N}}$$

Identity 4.35 (Permutation/Scatter Multiplicativity)**.**

$$\mathrm{perm}\left(\pi^{N \circlearrowleft} \right) \; \mathrm{S}_{w^{n \to N}} = \mathrm{S}_{(\pi^{-1} \circ w)^{n \to N}}$$

4.2.3 Gather/Scatter and Diagonals

Identity 4.36 (Commuting Gather with Diagonals).

$$\mathrm{G}_{r^{n \to N}} \ \mathrm{diag}\left(f^{N \to \mathbb{C}}\right) = \mathrm{diag}\left((f \circ r)^{n \to \mathbb{C}}\right) \mathrm{G}_{r^{n \to N}}$$

Identity 4.37 (Commuting Scatter with Diagonals).

$$\mathrm{diag}\left(f^{N \to \mathbb{C}}\right) \ \mathrm{S}_{w^{n \to N}} = \mathrm{S}_{w^{n \to N}} \ \mathrm{diag}\left((f \circ w)^{n \to \mathbb{C}}\right)$$

4.2.4 Iteration Reordering

Identity 4.38 (General Case).

$$\sum_{j=0}^{m-1} \mathrm{S}_{w_j} \, A_j \, \mathrm{G}_{r_j} = \sum_{k=0}^{m-1} \mathrm{S}_{w_{\pi(k)}} \, A_{\pi(k)} \, \mathrm{G}_{r_{\pi(k)}} \quad , \quad \pi \in \mathrm{S}_m$$

Identity 4.39 (Scatter Carried Reordering).

$$\sum_{j=0}^{m-1} \mathrm{S}_{(\pi \circ (j)_m) \otimes \imath_n} \, A_j \, \mathrm{G}_{r'_j} = \sum_{k=0}^{m-1} \mathrm{S}_{(k)_m \otimes \imath_n} \, A_{\pi^{-1}(k)} \, \mathrm{G}_{r'_{\pi^{-1}(k)}} \quad , \quad \pi \in \mathrm{S}_m$$

Identity 4.40 (Gather Carried Reordering).

$$\sum_{j=0}^{m-1} \mathrm{S}_{w'_j} \, A_j \, \mathrm{G}_{(\pi \circ (j)_m) \otimes \imath_n} = \sum_{k=0}^{m-1} \mathrm{S}_{w'_{\pi^{-1}(k)}} \, A_{\pi^{-1}(k)} \, \mathrm{G}_{(k)_m \otimes \imath_n} \quad , \quad \pi \in \mathrm{S}_m$$

4.2.5 Gather/Scatter and Inplace Computation

Identity 4.41.

$$\mathrm{S}_{w^{N_1 \to N}} \left(\sum_{j=0}^{m-1} \mathrm{S}_{f_j^{n \to N_1}} \, A_j \, \mathrm{G}_{f_j^{n \to N_1}} \right) = \left(\sum_{j=0}^{m-1} \mathrm{S}_{(w \circ f_j)^{n \to N}} \, A_j \, \mathrm{G}_{(w \circ f_j)^{n \to N}} \right) \mathrm{S}_{w^{N_1 \to N}}$$

Identity 4.42.

$$\left(\sum_{j=0}^{m-1} \mathrm{S}_{f_j^{n \to N_1}} \, A_j \, \mathrm{G}_{f_j^{n \to N_1}} \right) \mathrm{G}_{r^{N_1 \to N}} = \mathrm{G}_{r^{N_1 \to N}} \left(\sum_{j=0}^{m-1} \mathrm{S}_{(r \circ f_j)^{n \to N}} \, A_j \, \mathrm{G}_{(r \circ f_j)^{n \to N}} \right)$$

4.2.6 Gather/Scatter and Sums of Monomials

Identity 4.43 (Gather and Sum of Monomials).

$$\mathrm{G}_{r^{n \to N}} \left(\sum_{j=0}^{m-1} \mathrm{mon}\left(\pi_j^{N \circlearrowleft}, f_j^{N \to \mathbb{C}}\right) \right) = \sum_{j=0}^{m-1} \left(\mathrm{diag}\left((f_j \circ \pi_j \circ r)^{n \to \mathbb{C}}\right) \mathrm{G}_{(\pi_j \circ r)^{n \to N}} \right)$$

Identity 4.44 (Scatter and Sum of Monomials).

$$\left(\sum_{j=0}^{m-1} \mathrm{mon}\left(\pi_j^{N \circlearrowleft}, f_j^{N \to \mathbb{C}}\right) \right) \mathrm{S}_{w^{n \to N}} = \sum_{j=0}^{m-1} \left(\mathrm{S}_{\left(\pi_j^{-1} \circ w\right)^{n \to N}} \, \mathrm{diag}\left((f_j \circ w)^{n \to \mathbb{C}}\right) \right)$$

4.2.7 Sum-Accumulate to Standard Sum

Identity 4.45.

$$\begin{aligned}\mathrm{G}_{r^{n\to N}} + \mathrm{S}_{(k)_+^{n-m-k\to n}}\,\mathrm{G}_{s^{n-m-k\to N}} \quad = \quad & \mathrm{S}_{(0)_+^{k\to n}}\,\mathrm{G}_{r^{n\to N}\circ(0)_+^{k\to n}} + \\ & \mathrm{S}_{(k)_+^{n-m-k\to n}}\Big(\mathrm{G}_{r^{n\to N}\circ(k)_+^{n-k-m\to n}} + \mathrm{G}_{s^{n-k-m\to N}}\Big) + \\ & \mathrm{S}_{(n-m)_+^{m\to n}}\,\mathrm{G}_{r^{n\to N}\circ(n-m)_+^{m\to n}}\end{aligned}$$

4.3 Complex to Real

Identity 4.46 (Complex Number)**.**

$$\overline{a+ib} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad , \quad a,b\in\mathbb{R}$$

Identity 4.47 (Complex Matrix)**.**

$$\overline{\left[c_{i,j}\right]_{\substack{0\le i<m\\0\le j<n}}} = \left[\overline{c_{i,j}}\right]_{\substack{0\le i<m\\0\le j<n}} \quad , \quad c_{i,j}\in\mathbb{C}$$

Identity 4.48 (Diagonal Matrix)**.**

$$\overline{\mathrm{diag}\left(f^{n\to\mathbb{C}}\right)} = \mathrm{diag}\Big(\Re\big(f^{n\to\mathbb{C}}\big)\otimes \imath^{2\to\mathbb{C}}\Big) + \mathrm{mon}\Big(\imath_n\otimes \jmath_2, \Im\big(f^{n\to\mathbb{C}}\big)\otimes(\pm\imath)^{2\to\mathbb{C}}\Big)$$

Identity 4.49 (Product of Matrices)**.**

$$\overline{A\,B} = \overline{A}\,\overline{B}$$

Identity 4.50 (Matrix Transposition)**.**

$$\overline{A^\top} = \overline{A}^\top$$

Identity 4.51 (Sum of Matrices)**.**

$$\overline{\sum_{i=0}^{n-1} A_i} = \sum_{i=0}^{n-1}\overline{A_i}$$

Identity 4.52 (Parallel Tensor Product)**.**

$$\overline{\mathrm{I}_n\otimes A} = \mathrm{I}_n\otimes\overline{A}$$

Identity 4.53 (Vector Tensor Product)**.**

$$\overline{A^{k\times m}\otimes \mathrm{I}_n} = \left(\mathrm{I}_m\otimes \mathrm{L}_n^{2n}\right)\left(\overline{A^{k\times m}}\otimes\mathrm{I}_n\right)\left(\mathrm{I}_k\otimes\mathrm{L}_2^{2n}\right) \quad , \quad A^{k\times m}\in\mathbb{C}^{k\times m}$$

Identity 4.54 (Direct Sum of Matrices)**.**

$$\overline{\bigoplus_{i=0}^{n-1} A_i} = \bigoplus_{i=0}^{n-1}\overline{A_i}$$

Identity 4.55 (Real Matrix)**.**

$$\overline{A} = A\otimes\mathrm{I}_2 \quad , \quad A\in\mathbb{R}^{m\times n}$$

Identity 4.56 (Permutation Matrix)**.**

$$\overline{\mathrm{perm}(\pi)} = \mathrm{perm}(\pi\otimes\imath_2) \quad , \quad \pi\in\mathrm{S}_n$$

Identity 4.57 (Gather Matrix)**.**

$$\overline{\mathrm{G}}_r = \mathrm{G}_{r\otimes\imath_2}$$

Identity 4.58 (Scatter Matrix)**.**

$$\overline{\mathrm{S}}_w = \mathrm{S}_{w\otimes\imath_2}$$

4.4 Tensor Product Identities

Identity 4.59 (Tensor of Sums)**.**

$$\left(\sum_{j=0}^{m-1} A_j\right) \otimes \left(\sum_{k=0}^{n-1} B_k\right) = \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} A_j \otimes B_k$$

Identity 4.60 (Tensor of Products)**.**

$$\left(\prod_{j=0}^{n-1} A_j\right) \otimes \left(\prod_{j=0}^{n-1} B_j\right) = \prod_{j=0}^{n-1} A_j \otimes B_j$$

Identity 4.61.

$$A \otimes (B + C) = A \otimes B + A \otimes C$$

Identity 4.62.

$$(A + B) \oplus C = (A \oplus C) + (B \oplus 0_{m\times n}) \quad , \quad \text{for} \quad C \in \mathbb{C}^{m\times n}$$

Identity 4.63.

$$(\mathrm{I}_{n-1} \otimes A_m) \oplus 0_m = (\mathrm{I}_n \otimes A_m)(\mathrm{I}_{(n-1)m} \oplus 0_m)$$

Identity 4.64.

$$(A\,B)^P = A^P\,B^P \quad , \quad P = \mathrm{perm}(\pi)$$

Identity 4.65.

$$(A + B)^P = A^P + B^P \quad , \quad P = \mathrm{perm}(\pi)$$

Identity 4.66.

$$A^{r\times s} \otimes 0_{m\times n} = 0_{mr\times ns}$$

Identity 4.67.

$$0_{m\times n} \otimes A^{r\times s} = 0_{mr\times ns}$$

4.4.1 Rules for Conditional Matrices

Definition 74 (Conditional Matrix)**.**

$$\mathrm{Cond}(c, A, B) = \begin{cases} A & \text{if } c \\ B & \text{else} \end{cases}$$

Identity 4.68.

$$A + \mathrm{Cond}(c, B, C) = \mathrm{Cond}(c, A + B, A + C)$$

Identity 4.69.

$$\mathrm{Cond}(c, A, B) + C = \mathrm{Cond}(c, A + C, B + C)$$

Identity 4.70.

$$A\ \mathrm{Cond}(c, B, C) = \mathrm{Cond}(c, A\,B, A\,C)$$

Identity 4.71.

$$\mathrm{Cond}(c, A, B)\,C = \mathrm{Cond}(c, A\,C, B\,C)$$

Identity 4.72.

$$A \otimes \mathrm{Cond}(c, B, C) = \mathrm{Cond}(c, A \otimes B, A \otimes C)$$

Identity 4.73.

$$\mathrm{Cond}(c, A, B) \otimes C = \mathrm{Cond}(c, A \otimes C, B \otimes C)$$

4.5 Identities for Generated Matrices

4.5.1 Diagonal Matrix Identities

Identity 4.74.

$$\operatorname{diag}\left(c\imath^{n\to\mathbb{C}}\right) = c\operatorname{diag}\left(\imath^{n\to\mathbb{C}}\right)$$

4.5.2 Matrices as Monomials

Identity 4.75 (Zero Matrix as Monomial)**.**

$$0_n = \operatorname{mon}\left(\imath_n, (0)^{n\to\mathbb{C}}\right)$$

Identity 4.76 (Identity Matrix as Monomial)**.**

$$\mathrm{I}_n = \operatorname{mon}\left(\imath_n, (1)^{n\to\mathbb{C}}\right)$$

Identity 4.77 (Diagonal as Monomial)**.**

$$\operatorname{diag}\left(f^{n\to\mathbb{C}}\right) = \operatorname{mon}\left(\imath_n, f^{n\to\mathbb{C}}\right)$$

Identity 4.78 (Permutation as Monomial)**.**

$$\operatorname{perm}\left(\pi^{n\circlearrowleft}\right) = \operatorname{mon}\left(\pi^{n\circlearrowleft}, \mathrm{o}^{n\to\mathbb{C}}\right)$$

4.5.3 Matrix Operations on Generated Matrices

Identity 4.79 (Product of Permutations)**.**

$$\operatorname{perm}\left(\pi^{n\circlearrowleft}\right)\operatorname{perm}\left(\sigma^{n\circlearrowleft}\right) = \operatorname{perm}\left((\sigma\circ\pi)^{n\circlearrowleft}\right)$$

Identity 4.80 (Product of Diagonals)**.**

$$\operatorname{diag}\left(f^{n\to\mathbb{C}}\right)\operatorname{diag}\left(g^{n\to\mathbb{C}}\right) = \operatorname{diag}\left((fg)^{n\to\mathbb{C}}\right)$$

Identity 4.81 (Product of Monomials)**.**

$$\operatorname{mon}\left(\pi^{n\circlearrowleft}, f^{n\to\mathbb{C}}\right)\operatorname{mon}\left(\sigma^{n\circlearrowleft}, g^{n\to\mathbb{C}}\right) = \operatorname{mon}\left(\sigma\circ\pi, (f\circ\sigma^{-1})g\right)$$

Identity 4.82 (Tensor Product of Permutations)**.**

$$\operatorname{perm}\left(\pi^{m\circlearrowleft}\right)\otimes\operatorname{perm}\left(\sigma^{n\circlearrowleft}\right) = \operatorname{perm}\left((\pi\otimes\sigma)^{mn\circlearrowleft}\right)$$

Identity 4.83 (Tensor Product of Diagonals)**.**

$$\operatorname{diag}\left(f^{m\to\mathbb{C}}\right)\otimes\operatorname{diag}\left(g^{n\to\mathbb{C}}\right) = \operatorname{diag}\left((f\otimes g)^{mn\to\mathbb{C}}\right)$$

Identity 4.84 (Tensor Product of Monomials)**.**

$$\operatorname{mon}\left(\pi^{n\circlearrowleft}, f^{n\to\mathbb{C}}\right)\otimes\operatorname{mon}\left(\sigma^{n\circlearrowleft}, g^{n\to\mathbb{C}}\right) = \operatorname{mon}\left(\pi\otimes\sigma, f\otimes g\right)$$

Identity 4.85 (Conjugation of Diagonals)**.**

$$\operatorname{diag}\left(f^{n\to\mathbb{C}}\right)^{\operatorname{perm}\left(\pi^{n\circlearrowleft}\right)} = \operatorname{diag}\left(\left(f\circ\pi^{-1}\right)^{n\to\mathbb{C}}\right)$$

Identity 4.86 (Conjugation of Permutations)**.**

$$\operatorname{perm}\left(\pi^{n\circlearrowleft}\right)^{\operatorname{perm}\left(\sigma^{n\circlearrowleft}\right)} = \operatorname{perm}\left((\pi^{\sigma})^{n\circlearrowleft}\right)$$

Identity 4.87 (Conjugation of Monomial)**.**

$$\operatorname{mon}\left(\pi^{n\circlearrowleft}, f^{n\to\mathbb{C}}\right)^{\operatorname{perm}\left(\sigma^{n\circlearrowleft}\right)} = \operatorname{mon}\left((\pi^{\sigma})^{n\circlearrowleft}, \left(f\circ\sigma^{-1}\right)^{n\to\mathbb{C}}\right)$$

Identity 4.88 (Commuting Diagonal and Permutation)**.**

$$\operatorname{diag}\left(f^{n\to\mathbb{C}}\right)\operatorname{perm}\left(\pi^{n\circlearrowleft}\right) = \operatorname{perm}\left(\pi^{n\to n}\right)\operatorname{diag}\left(\left(f\circ\pi^{-1}\right)^{n\to\mathbb{C}}\right)$$

4.5.4 Special Identities

Identity 4.89 (Maintaining Tensor Structure).

$$\text{mon}\left(\imath_m \otimes \pi^{n\circlearrowleft}, f^{mn\to\mathbb{C}}\right) \oplus 0_{rn} = \text{mon}\left(\imath_{m+r} \otimes \pi^{n\circlearrowleft}, f^{mn\to\mathbb{C}} \oplus (0)^{rn\to\mathbb{C}}\right)$$

Identity 4.90 (Maintaining Tensor Structure).

$$0_{rn} \oplus \text{mon}\left(\imath_m \otimes \pi^{n\circlearrowleft}, f^{mn\to\mathbb{C}}\right) = \text{mon}\left(\imath_{r+m} \otimes \pi^{n\circlearrowleft}, (0)^{rn\to\mathbb{C}} \oplus f^{mn\to\mathbb{C}}\right)$$

Identity 4.91 (Maintaining Tensor Structure).

$$\text{mon}\left(\imath_m \otimes \pi^{n\circlearrowleft}, \imath^{m\to\mathbb{C}} \otimes f^{n\to\mathbb{C}}\right) \oplus 0_{rn} = \text{mon}\left(\imath_{m+r} \otimes \pi^{n\circlearrowleft}, \delta^{m+r}_{\mathbb{I}_m} \otimes f^{n\to\mathbb{C}}\right)$$

Identity 4.92 (Maintaining Tensor Structure).

$$0_{rn} \oplus \text{mon}\left(\imath_m \otimes \pi^{n\circlearrowleft}, \imath^{m\to\mathbb{C}} \otimes f^{n\to\mathbb{C}}\right) = \text{mon}\left(\imath_{m+r} \otimes \pi^{n\circlearrowleft}, \delta^{m+r}_{(r)_+^{m\to m+r}(\mathbb{I}_m)} \otimes f^{n\to\mathbb{C}}\right)$$

4.5.5 Delta-Diagonals and Gathers

Identity 4.93.

$$\text{diag}\left(f^{m\to\mathbb{C}} \otimes g^{n\to\mathbb{C}}\right) \text{G}_{r^{m\to M} \otimes s^{n\to N}} = \Big(\text{diag}\left(f^{m\to\mathbb{C}}\right) \text{G}_{r^{m\to M}}\Big) \otimes \Big(\text{diag}\left(g^{n\to\mathbb{C}}\right) \text{G}_{s^{n\to N}}\Big)$$

Identity 4.94.

$$\text{diag}\left(\delta^m_{\mathbb{I}_{k,r}}\right) = \text{S}_{(k)_+^{r-k\to m}} \, \text{G}_{(k)_+^{r-k\to m}}$$

Identity 4.95.

$$\text{diag}\left(\delta^m_{\mathbb{I}_{k,n}}\right) \text{G}_{\text{z}^r_m} = \text{S}_{(k)_+^{n-k\to m}} \, \text{G}_{((k+r) \bmod m)_+^{n-k\to m}} \quad , \quad \text{for } r \le m-n \text{ or } r \ge m-k$$

Identity 4.96.

$$\text{diag}\left(\delta^n_N \circ (j)_n\right) \text{G}_{(j)_n} = \text{Cond}\Big(j \in N, \text{G}_{(j)_n}, 0_{1\times n}\Big)$$

Identity 4.97.

$$\text{diag}\left(\delta^n_{\mathbb{I}_{n-k}} \circ (j)_n\right) \text{G}_{(\pi^{n-k\circlearrowleft} \oplus \sigma^{k\circlearrowleft}) \circ (j)_n} = \text{Cond}\Big(j \in \mathbb{I}_{n-k}, \text{G}_{(0)_+^{n-k\to n} \circ \pi^{n-k\circlearrowleft} \circ (j)_{n-k}}, 0_{1\times n}\Big)$$

Identity 4.98.

$$\text{diag}\left(\delta^n_{\mathbb{I}_{k,n}} \circ (j)_n\right) \text{G}_{(\pi^{k\circlearrowleft} \oplus \sigma^{n-k\circlearrowleft}) \circ (j)_n} = \text{Cond}\Big(j \in \mathbb{I}_{k,n}, \text{G}_{(k)_+^{n-k\to n} \circ \sigma^{n-k\circlearrowleft} \circ (j-k)_{n-k}}, 0_{1\times n}\Big)$$

4.5.6 Delta-Diagonals and Scatter

Identity 4.99.

$$\text{S}_{v^{m\to M} \otimes w^{n\to N}} \, \text{diag}\left(f^{m\to\mathbb{C}} \otimes g^{n\to\mathbb{C}}\right) = \Big(\text{S}_{v^{m\to M}} \, \text{diag}\left(f^{m\to\mathbb{C}}\right)\Big) \otimes \Big(\text{S}_{w^{n\to N}} \, \text{diag}\left(g^{n\to\mathbb{C}}\right)\Big)$$

Identity 4.100.

$$\text{S}_{\text{z}^k_m} \, \text{diag}\left(\delta^m_{\mathbb{I}_{m-k}}\right) = \text{S}_{(k)_+^{m-k\to m}} \, \text{G}_{(0)_+^{m-k\to m}}$$

Identity 4.101.

$$\text{S}_{\text{z}^{m-k}_m} \, \text{diag}\left(\delta^m_{\mathbb{I}_{k,m}}\right) = \text{S}_{(0)_+^{m-k\to m}} \, \text{G}_{(k)_+^{m-k\to m}}$$

Identity 4.102.

$$\text{S}_{(j)_n} \, \text{diag}\left(\delta^n_N \circ (j)_n\right) = \text{Cond}\Big(j \in N, \text{S}_{(j)_n}, 0_{n\times 1}\Big)$$

Identity 4.103.

$$\text{S}_{(\pi^{n-k\circlearrowleft} \oplus \sigma^{k\circlearrowleft}) \circ (j)_n} \, \text{diag}\left(\delta^n_{\mathbb{I}_{n-k}} \circ (j)_n\right) = \text{Cond}\Big(j \in \mathbb{I}_{n-k}, \text{S}_{(0)_+^{n-k\to n} \circ \pi^{n-k\circlearrowleft} \circ (j)_{n-k}}, 0_{n\times 1}\Big)$$

Identity 4.104.

$$\text{S}_{(\pi^{k\circlearrowleft} \oplus \sigma^{n-k\circlearrowleft}) \circ (j)_n} \, \text{diag}\left(\delta^n_{\mathbb{I}_{k,n}} \circ (j)_n\right) = \text{Cond}\Big(j \in \mathbb{I}_{k,n}, \text{S}_{(k)_+^{n-k\to n} \circ \sigma^{n-k\circlearrowleft} \circ (j-k)_{n-k}}, 0_{n\times 1}\Big)$$

4.5.7 Matrix Structure

Property 4.1 (Cyclic Shift).

$$\mathrm{Z}_n^k = \begin{bmatrix} & \mathrm{I}_{n-k} \\ \mathrm{I}_k & \end{bmatrix}$$

Definition 75 (Upper Diagonal Matrix).

$$\mathrm{U}_n^k = \mathrm{Z}_n^k \left(0_{k\times k} \oplus \mathrm{I}_{n-k} \right)$$

Property 4.2 (Upper Diagonal Matrix).

$$\mathrm{U}_n^k = \mathrm{mon}\left(z_n^k, \delta_{\mathbb{I}_{k,n}}^n \right)$$

Definition 76 (Lower Diagonal Matrix).

$$\mathrm{H}_n^k = \mathrm{Z}_n^{n-k} \left(\mathrm{I}_{n-k} \oplus 0_{k\times k} \right)$$

Property 4.3 (Lower Diagonal Matrix).

$$\mathrm{H}_n^k = \mathrm{mon}\left(z_n^{n-k}, \delta_{\mathbb{I}_{n-k}}^n \right)$$

Definition 77 (S Matrix).

$$\mathrm{S}_n = \mathrm{I}_n + \mathrm{U}_n^1$$

Property 4.4 (S Matrix).

$$\mathrm{S}_n = \begin{bmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix} \quad , \quad \mathrm{S}_n \in \mathbb{C}^{n\times n}$$

Property 4.5 (Transposed S Matrix).

$$\mathrm{S}_n^\top = \mathrm{I}_n + \mathrm{H}_n^1$$

Property 4.6.

$$\mathrm{J}_{n-1} \oplus 0_{1\times 1} = \mathrm{mon}\left(\mathrm{z}_n^1 \circ \jmath_n, \delta_{0,\ldots,n-2}^n \right)$$

Property 4.7.

$$0_{1\times 1} \oplus \mathrm{J}_{n-1} = \mathrm{mon}\left(\jmath_n \circ \mathrm{z}_n^1, \delta_{1,\ldots,n-1}^n \right)$$

5 Cooley-Tukey Algorithms

5.1 Discrete Fourier Transform

Theorem 5.1 (DFT DIT Recursion).

$$\mathrm{DFT}_{mn} = \left(\mathrm{DFT}_m \otimes \mathrm{I}_n \right) \mathrm{T}_n^{mn} \left(\mathrm{I}_m \otimes \mathrm{DFT}_n \right) \mathrm{L}_m^{mn}$$

Theorem 5.2 (DFT DIF Recursion).

$$\mathrm{DFT}_{mn} = \mathrm{L}_n^{mn} \left(\mathrm{I}_m \otimes \mathrm{DFT}_n \right) \mathrm{T}_n^{mn} \left(\mathrm{DFT}_m \otimes \mathrm{I}_n \right)$$

Theorem 5.3 (2D DFT DIT Vector Radix Recursion).

$$\mathrm{DFT}_{mn\times rs} = \left(\mathrm{DFT}_{m\times r} \otimes \mathrm{I}_{ns} \right)^{\mathrm{I}_m \otimes \mathrm{L}_r^{rn} \otimes \mathrm{I}_s} \left(\mathrm{T}_n^{mn} \otimes \mathrm{T}_s^{rs} \right) \left(\mathrm{I}_{mr} \otimes \mathrm{DFT}_{n\times s} \right)^{\mathrm{I}_m \otimes \mathrm{L}_r^{rn} \otimes \mathrm{I}_s} \left(\mathrm{L}_m^{mn} \otimes \mathrm{L}_r^{rs} \right)$$

5.2 Discrete Trigonometric Transforms

Definition 78 (Zeros Recursion).

$$\mathrm{r}_i^m : \begin{cases} \mathbb{R} \to \mathbb{R} \\ r \mapsto \begin{cases} \frac{r+2\lfloor \frac{i}{2} \rfloor}{m} & i \bmod 2 = 0 \\ \frac{2-r+2\lfloor \frac{i}{2} \rfloor}{m} & i \bmod 2 = 1 \end{cases} \end{cases}$$

Theorem 5.4 (U-Basis DIF Recursion for DTT).

$$\mathrm{DTT}_{mn}(r) = \mathrm{K}_n^{mn} \left(\bigoplus_{i=0}^{m-1} \mathrm{DTT}_n \left(\mathrm{r}_i^m(r) \right) \right) \left(\overline{\mathrm{DST}\text{-}3}_m(r) \otimes \mathrm{I}_n \right) B_{mn,m}^{\mathrm{DTT}/U}$$

Theorem 5.5 (U-Basis DIF Recursion for $\overline{\mathrm{DTT}}$).

$$\overline{\mathrm{DTT}}_{mn}(r) = \mathrm{K}_n^{mn} \left(\bigoplus_{i=0}^{m-1} \overline{\mathrm{DTT}}_n \left(\mathrm{r}_i^m(r) \right) \right) \left(\overline{\mathrm{DST}\text{-}3}_m(r) \otimes \mathrm{I}_n \right) B_{mn,m}^{\mathrm{DTT}/U}$$

Theorem 5.6 (U-Basis DIT Recursion for DTT).

$$\mathrm{DTT}_{mn}(r) = B_{mn,m}^{\top,\mathrm{DTT}/U} \left(\overline{\mathrm{DST}\text{-}2}_m(r) \otimes \mathrm{I}_n \right) \left(\bigoplus_{i=0}^{m-1} \mathrm{DTT}_n \left(\mathrm{r}_i^m(r) \right) \right) \mathrm{M}_m^{mn}$$

Theorem 5.7 (U-Basis DIT Recursion for $\overline{\mathrm{DTT}}$).

$$\overline{\mathrm{DTT}}_{mn}(r) = B_{mn,m}^{\top,\mathrm{DTT}/U} \left(\overline{\mathrm{DST}\text{-}2}_m(r) \otimes \mathrm{I}_n \right) \left(\bigoplus_{i=0}^{m-1} \overline{\mathrm{DTT}}_n \left(\mathrm{r}_i^m(r) \right) \right) \mathrm{M}_m^{mn}$$

Theorem 5.8 (T-Basis DIF Recursion for DTT).

$$\mathrm{DTT}_{mn}(r) = \mathrm{K}_n^{mn} \left(\bigoplus_{i=0}^{m-1} \mathrm{DTT}_n \left(\mathrm{r}_i^m(r) \right) \right) \left(\mathrm{DCT}\text{-}3_m(r) \otimes \mathrm{I}_n \right) B_{mn,m}^{\mathrm{DTT}/T}$$

Theorem 5.9 (T-Basis DIF Recursion for $\overline{\mathrm{DTT}}$).

$$\overline{\mathrm{DTT}}_{mn}(r) = \mathrm{K}_n^{mn} \left(\bigoplus_{i=0}^{m-1} \overline{\mathrm{DTT}}_n \left(\mathrm{r}_i^m(r) \right) \right) \left(\mathrm{DCT}\text{-}3_m(r) \otimes \mathrm{I}_n \right) B_{mn,m}^{\mathrm{DTT}/T}$$

Theorem 5.10 (T-Basis DIT Recursion for DTT).

$$\mathrm{DTT}_{mn}(r) = B_{mn,m}^{\top,\mathrm{DTT}/T} \left(\mathrm{DCT}\text{-}2_m(r) \otimes \mathrm{I}_n \right) \left(\bigoplus_{i=0}^{m-1} \mathrm{DTT}_n \left(\mathrm{r}_i^m(r) \right) \right) \mathrm{M}_m^{mn}$$

Theorem 5.11 (T-Basis DIT Recursion for $\overline{\mathrm{DTT}}$).

$$\overline{\mathrm{DTT}}_{mn}(r) = B_{mn,m}^{\top,\mathrm{DTT}/T} \left(\mathrm{DCT}\text{-}2_m(r) \otimes \mathrm{I}_n \right) \left(\bigoplus_{i=0}^{m-1} \overline{\mathrm{DTT}}_n \left(\mathrm{r}_i^m(r) \right) \right) \mathrm{M}_m^{mn}$$

Theorem 5.12 (T-Basis DIT Recursion for iDTT).

$$\mathrm{iDTT}_{mn}(r) = C_{mn,m}^{-1,\mathrm{DTT}/T} \left(\mathrm{iDCT}\text{-}3_m(r) \otimes \mathrm{I}_n \right) \left(\bigoplus_{i=0}^{m-1} \mathrm{iDTT}_n \left(\mathrm{r}_i^m(r) \right) \right) \mathrm{M}_m^{mn}$$

5.3 DIF Rules for DTTs

5.3.1 DST-3, U-Basis

Definition 79 (Base Change DST-3, U-Basis).

$$B^{\mathrm{DST\text{-}3}/U}_{mn,m} = \big((\mathrm{I}_{n-1} \otimes \mathrm{S}_m) \oplus \mathrm{I}_m \big)\, \mathrm{L}^{mn}_n \big(\underbrace{(\mathrm{I}_{n-1} \oplus \mathrm{I}_1) \oplus (\mathrm{J}_{n-1} \oplus \mathrm{I}_1) \oplus \cdots}_{m \text{ summands}} \big)$$

Property 5.1 (Base Change DST-3, U-Basis).

$$B^{\mathrm{DST\text{-}3}/U}_{mn,m} = \begin{bmatrix} \mathrm{I}_{n-1} & & \mathrm{J}_{n-1} & & & & & & \\ & 1 & & & & & & & \\ & & \mathrm{I}_{n-1} & & \mathrm{J}_{n-1} & & & & \\ & & & 1 & & & & & \\ & & & & \ddots & & \ddots & & \\ & & & & & \mathrm{I}_{n-1} & & \mathrm{J}_{n-1} & \\ & & & & & & 1 & & \\ & & & & & & & \mathrm{I}_{n-1} & \\ & & & & & & & & 1 \end{bmatrix}$$

Property 5.2 (Base Change DST-3, U-Basis).

$$B^{\mathrm{DST\text{-}3}/U}_{mn,m} = \mathrm{I}_{mn} + \mathrm{U}^1_m \otimes (\mathrm{J}_{n-1} \oplus 0_{1\times 1})$$

Property 5.3 (Base Change DST-3, U-Basis).

$$B^{\mathrm{DST\text{-}3}/U}_{mn,m} = \mathrm{diag}\left(\imath^{mn\to\mathbb{C}}\right) + \mathrm{mon}\left(\mathrm{z}^1_m \otimes (\jmath_{n-1} \oplus \imath_1), \delta^m_{\mathbb{I}_{1,m}} \otimes \delta^n_{\mathbb{I}_{0,n-1}}\right)$$

Theorem 5.13 (Base Case DST-3).

$$\begin{aligned} \mathrm{DST\text{-}3}_2(r) &= \mathrm{F}_2 \ \mathrm{diag}\left(\sin \tfrac{r\pi}{2}, \sin r\pi\right) \\ \overline{\mathrm{DST\text{-}3}}_2(r) &= \mathrm{F}_2 \ \mathrm{diag}\left(1, 2\cos \tfrac{r\pi}{2}\right) \end{aligned}$$

Theorem 5.14 (Recursion DST-3).

$$\begin{aligned} \mathrm{DST\text{-}3}_n &= \mathrm{DST\text{-}3}_n(1/2) \\ \mathrm{DST\text{-}3}_{mn}(r) &= \mathrm{K}^{mn}_n \left(\bigoplus_{i=0}^{m-1} \mathrm{DST\text{-}3}_n\left(\mathrm{r}^m_i(r)\right) \right) \left(\overline{\mathrm{DST\text{-}3}}_m(r) \otimes \mathrm{I}_n \right) B^{\mathrm{DST\text{-}3}/U}_{mn,m} \\ \overline{\mathrm{DST\text{-}3}}_{mn}(r) &= \mathrm{K}^{mn}_n \left(\bigoplus_{i=0}^{m-1} \overline{\mathrm{DST\text{-}3}}_n\left(\mathrm{r}^m_i(r)\right) \right) \left(\overline{\mathrm{DST\text{-}3}}_m(r) \otimes \mathrm{I}_n \right) B^{\mathrm{DST\text{-}3}/U}_{mn,m} \end{aligned}$$

5.3.2 DCT-3, U-Basis

Definition 80 (Base Change DCT-3, U-Basis).

$$B^{\mathrm{DCT\text{-}3}/U}_{mn,m} = \left(\tfrac{1}{2}\left(2\,\mathrm{I}_1 \oplus \mathrm{I}_{m-1} - \mathrm{U}^2_m\right) \oplus \left(\mathrm{I}_{n-1} \otimes \mathrm{S}_m\right) \right) \mathrm{L}^{mn}_n \big(\underbrace{(\mathrm{I}_1 \oplus \mathrm{I}_{n-1}) \oplus (\mathrm{I}_1 \oplus \mathrm{J}_{n-1}) \oplus \cdots}_{m \text{ summands}} \big)$$

Property 5.4 (Base Change DCT-3, U-Basis).

$$B^{\text{DCT-3}/U}_{mn,m} = \left[\begin{array}{cccccccccccc}
1 & & & & -\frac{1}{2} & & & & & & & \\
 & \mathrm{I}_{n-1} & & \mathrm{J}_{n-1} & & & & & & & & \\
 & & \frac{1}{2} & & & & -\frac{1}{2} & & & & & \\
 & & & \mathrm{I}_{n-1} & & \mathrm{J}_{n-1} & & \ddots & & & & \\
 & & & & \ddots & & \ddots & & \ddots & & & \\
 & & & & & \ddots & & \ddots & & \ddots & & \\
 & & & & & & \frac{1}{2} & & \ddots & & -\frac{1}{2} & \\
 & & & & & & & \mathrm{I}_{n-1} & & \mathrm{J}_{n-1} & & \\
 & & & & & & & & \frac{1}{2} & & & \\
 & & & & & & & & & \mathrm{I}_{n-1} & & \mathrm{J}_{n-1} \\
 & & & & & & & & & & \frac{1}{2} & \\
 & & & & & & & & & & & \mathrm{I}_{n-1}
\end{array}\right]$$

Property 5.5 (Base Change DCT-3, U-Basis).

$$B^{\text{DCT-3}/U}_{mn,m} = \mathrm{I}_n \oplus \big(\mathrm{I}_{m-1} \otimes (\tfrac{1}{2} \oplus \mathrm{I}_{n-1})\big) + \mathrm{U}^1_m \otimes (0_{1\times 1} \oplus \mathrm{J}_{n-1}) - \mathrm{U}^2_m \otimes (\tfrac{1}{2} \oplus 0_{n-1\times n-1})$$

Property 5.6 (Base Change DCT-3, U-Basis).

$$\begin{aligned}
B^{\text{DCT-3}/U}_{mn,m} \quad = \quad & \operatorname{diag}\Big(\big(\delta^m_{\mathbb{I}_1} \otimes \delta^n_{\mathbb{I}_1}\big) + \tfrac{1}{2}\big(\delta^m_{\mathbb{I}_{1,m}} \otimes \delta^n_{\mathbb{I}_1}\big) + \big(\delta^m_{\mathbb{I}_m} \otimes \delta^n_{\mathbb{I}_{1,n}}\big)\Big) + \\
& + \operatorname{mon}\Big(\mathrm{z}^1_m \otimes (\imath_1 \oplus \jmath_{n-1}), \delta^m_{\mathbb{I}_{1,m}} \otimes \delta^n_{\mathbb{I}_{1,n}}\Big) + \\
& + \operatorname{mon}\Big(\mathrm{z}^2_m \otimes \imath_n, -\tfrac{1}{2}\big(\delta^m_{\mathbb{I}_{2,m}} \otimes \delta^n_{\mathbb{I}_1}\big)\Big)
\end{aligned}$$

Theorem 5.15 (Base Case DCT-3).

$$\text{DCT-3}_2(r) \quad = \quad \mathrm{F}_2 \operatorname{diag}(1, \cos r\pi)$$

Theorem 5.16 (Recursion DCT-3).

$$\begin{aligned}
\text{DCT-3}_n \quad & = \quad \text{DCT-3}_n(1/2) \\
\text{DCT-3}_{mn}(r) \quad & = \quad \mathrm{K}^{mn}_n \left(\bigoplus_{i=0}^{m-1} \text{DCT-3}_n\big(\mathrm{r}^m_i(r)\big)\right) \Big(\overline{\text{DST-3}}_m(r) \otimes \mathrm{I}_n\Big) B^{\text{DCT-3}/U}_{mn,m}
\end{aligned}$$

5.3.3 DST-4, U-Basis

Definition 81 (Base Change DST-4, U-Basis).

$$B^{\text{DST-4}/U}_{mn,m} = \big(\mathrm{S}_m \otimes \mathrm{I}_n\big)^{\overbrace{\mathrm{I}_n \oplus \mathrm{J}_n \oplus \cdots}^{}_{m \text{ summands}}}$$

Property 5.7 (Base Change DST-4, U-Basis).

$$B^{\text{DST-4}/U}_{mn,m} = \begin{bmatrix}
\mathrm{I}_n & \mathrm{J}_n & & & \\
 & \mathrm{I}_n & \mathrm{J}_n & & \\
 & & \ddots & \ddots & \\
 & & & \mathrm{I}_n & \mathrm{J}_n \\
 & & & & \mathrm{I}_n
\end{bmatrix}$$

Property 5.8 (Base Change DST-4, U-Basis).

$$B^{\text{DST-4}/U}_{mn,m} = \mathrm{I}_{mn} + \mathrm{U}^1_m \otimes \mathrm{J}_n$$

Property 5.9 (Base Change DST-4, U-Basis).

$$B_{mn,m}^{\mathrm{DST\text{-}4}/U} = \mathrm{diag}\left(\imath^{mn\to\mathbb{C}}\right) + \mathrm{mon}\left(\mathrm{z}_m^1 \otimes \jmath_n, \delta_{\mathbb{I}_{1,m}}^m \otimes \imath^{n\to\mathbb{C}}\right)$$

Theorem 5.17 (Base Case DST-4).

$$\mathrm{DST\text{-}4}_2(r) = \mathrm{diag}\left(\sin\tfrac{r\pi}{4}, \cos\tfrac{r\pi}{4}\right)\mathrm{F}_2 \begin{bmatrix} 1 & 1 \\ 0 & 2\cos\tfrac{r\pi}{2} \end{bmatrix}$$

Theorem 5.18 (Recursion DST-4).

$$\begin{aligned} \mathrm{DST\text{-}4}_n &= \mathrm{DST\text{-}4}_n(1/2) \\ \mathrm{DST\text{-}4}_{mn}(r) &= \mathrm{K}_n^{mn}\left(\bigoplus_{i=0}^{m-1}\mathrm{DST\text{-}4}_n\left(\mathrm{r}_i^m(r)\right)\right)\left(\overline{\mathrm{DST\text{-}3}}_m(r)\otimes \mathrm{I}_n\right)B_{mn,m}^{\mathrm{DST\text{-}4}/U} \end{aligned}$$

5.3.4 DCT-4, U-Basis

Definition 82 (Base Change DCT-4, U-Basis).

$$B_{mn,m}^{\mathrm{DCT\text{-}4}/U} = \left((\mathrm{I}_m - \mathrm{U}_m^1)\otimes \mathrm{I}_n\right)\underbrace{\mathrm{I}_n\oplus \mathrm{J}_n\oplus\cdots}_{m\text{ summands}}$$

Property 5.10 (Base Change DCT-4, U-Basis).

$$B_{mn,m}^{\mathrm{DCT\text{-}4}/U} = \begin{bmatrix} \mathrm{I}_n & -\mathrm{J}_n & & & \\ & \mathrm{I}_n & -\mathrm{J}_n & & \\ & & \ddots & \ddots & \\ & & & \mathrm{I}_n & -\mathrm{J}_n \\ & & & & \mathrm{I}_n \end{bmatrix}$$

Property 5.11 (Base Change DCT-4, U-Basis).

$$B_{mn,m}^{\mathrm{DCT\text{-}4}/U} = \mathrm{I}_{mn} - \mathrm{U}_m^1 \otimes \mathrm{J}_n$$

Property 5.12 (Base Change DCT-4, U-Basis).

$$B_{mn,m}^{\mathrm{DCT\text{-}4}/U} = \mathrm{diag}\left(\imath^{mn\to\mathbb{C}}\right) + \mathrm{mon}\left(\mathrm{z}_m^1 \otimes \jmath_n, \delta_{\mathbb{I}_{1,m}}^m \otimes \left(-\imath^{n\to\mathbb{C}}\right)\right)$$

Theorem 5.19 (Base Case DCT-4).

$$\mathrm{DCT\text{-}4}_2(r) = \mathrm{diag}\left(\cos\tfrac{r\pi}{4}, \sin\tfrac{r\pi}{4}\right)\mathrm{F}_2 \begin{bmatrix} 1 & -1 \\ 0 & 2\cos\tfrac{r\pi}{2} \end{bmatrix}$$

Theorem 5.20 (Recursion DCT-4).

$$\begin{aligned} \mathrm{DCT\text{-}4}_n &= \mathrm{DCT\text{-}4}_n(1/2) \\ \mathrm{DCT\text{-}4}_{mn}(r) &= \mathrm{K}_n^{mn}\left(\bigoplus_{i=0}^{m-1}\mathrm{DCT\text{-}4}_n\left(\mathrm{r}_i^m(r)\right)\right)\left(\overline{\mathrm{DST\text{-}3}}_m(r)\otimes \mathrm{I}_n\right)B_{mn,m}^{\mathrm{DCT\text{-}4}/U} \end{aligned}$$

5.3.5 DCT-3, T-Basis

Definition 83 (Base Change Transposed iDCT-3, T-Basis).

$$C_{mn,m}^{-\top,\mathrm{DCT\text{-}3}/T} = \left(\mathrm{I}_m \oplus \left(\mathrm{I}_{n-1}\otimes \mathrm{S}_m^\top\right)\right)\mathrm{L}_n^{mn}\left(\underbrace{(\mathrm{I}_1\oplus \mathrm{I}_{n-1})\oplus(\mathrm{I}_1\oplus \mathrm{J}_{n-1})\oplus\cdots}_{m\text{ summands}}\right)$$

Property 5.13 (Base Change Transposed iDCT-3, T-Basis).

$$C_{mn,m}^{-\top,\mathrm{DCT\text{-}3}/T} = \begin{bmatrix} 1 & & & & & & & & & \\ & \mathrm{I}_{n-1} & & & & & & & & \\ & & 1 & & & & & & & \\ & \mathrm{J}_{n-1} & & \mathrm{I}_{n-1} & & & & & & \\ & & \ddots & & \ddots & & & & & \\ & & & \ddots & & \ddots & & & & \\ & & & & \ddots & & 1 & & & \\ & & & & & \mathrm{J}_{n-1} & & \mathrm{I}_{n-1} & & \\ & & & & & & & & 1 & \\ & & & & & & & \mathrm{J}_{n-1} & & \mathrm{I}_{n-1} \end{bmatrix}$$

Property 5.14 (Base Change Transposed iDCT-3, T-Basis).

$$C_{mn,m}^{-\top,\mathrm{DCT\text{-}3}/T} = \mathrm{I}_{mn} + \mathrm{H}_m^1 \otimes (0_{1\times 1} \oplus \mathrm{J}_{n-1})$$

Property 5.15 (Base Change Transposed iDCT-3, T-Basis).

$$C_{mn,m}^{-\top,\mathrm{DCT\text{-}3}/T} = \mathrm{diag}\left(\imath^{mn\to\mathbb{C}}\right) + \mathrm{mon}\left(\mathrm{z}_m^{m-1} \otimes (\imath_1 \oplus \jmath_{n-1}), \delta_{\mathbb{I}_{m-1}}^m \otimes \delta_{\mathbb{I}_{1,n}}^n\right)$$

Theorem 5.21 (Recursion DCT-3).

$$\begin{aligned} \mathrm{DCT\text{-}3}_n &= \mathrm{iDCT\text{-}3}_n(1/2)^\top \\ \mathrm{iDCT\text{-}3}_{mn}(r)^\top &= \mathrm{K}_n^{mn}\left(\bigoplus_{i=0}^{m-1} \mathrm{iDCT\text{-}3}_n\left(\mathrm{r}_i^m(r)\right)^\top\right)\left(\mathrm{iDCT\text{-}3}_m(r)^\top \otimes \mathrm{I}_n\right) C_{mn,m}^{-\top,\mathrm{DCT\text{-}3}/T} \end{aligned}$$

5.3.6 DCT-3, T-Basis, Radix-2

Definition 84 (Base Change DCT-3, T-Basis, Radix-2).

$$C_{2n,2}^{\mathrm{DCT\text{-}3}/T} = \left(\mathrm{I}_2 \oplus \left(\mathrm{I}_{n-1} \otimes (\mathrm{I}_2 - \mathrm{U}_2^1)\right)\right)^{\mathrm{L}_n^{2n}} \left(\mathrm{I}_n \oplus (\mathrm{I}_1 \oplus \mathrm{J}_{n-1})\right)$$

Property 5.16 (Base Change DCT-3, T-Basis, Radix-2).

$$C_{2n,2}^{\mathrm{DCT\text{-}3}/T} = \mathrm{I}_{2n} - \mathrm{U}_2^1 \otimes (0_1 \oplus \mathrm{J}_{n-1})$$

Definition 85 (Base Change Diagonal DCT-3, T-Basis, Radix-2).

$$E_{2n,2}^{\mathrm{DCT\text{-}3}/T,r} = \mathrm{I}_n \oplus \left(\cos r\pi\ \mathrm{diag}\left(\mathrm{I}_1 \oplus 2\,\mathrm{I}_{n-1}\right)\right)$$

Property 5.17 (Base Change DCT-3, T-Basis, Radix-2).

$$B_{2n,2}^{\mathrm{DCT\text{-}3}/T,r} = \begin{bmatrix} \mathrm{I}_1 & & & \\ & \mathrm{I}_{n-1} & & -\mathrm{J}_{n-1} \\ & & \cos r\pi\,\mathrm{I}_1 & \\ & & & 2\cos r\pi\,\mathrm{I}_{n-1} \end{bmatrix}$$

Property 5.18 (Base Change DCT-3, T-Basis, Radix-2).

$$\begin{aligned} B_{2n,2}^{\mathrm{DCT\text{-}3}/T,r} &= \mathrm{diag}\left(\left(\delta_{\mathbb{I}_1}^2 \otimes \imath^{n\to\mathbb{C}}\right) + \cos r\pi\left(\delta_{\mathbb{I}_{1,2}}^2 \otimes \delta_{\mathbb{I}_1}^n\right) + 2\cos r\pi\left(\delta_{\mathbb{I}_{1,2}}^2 \otimes \delta_{\mathbb{I}_{1,n}}^n\right)\right) \\ &\quad + \mathrm{mon}\left(\mathrm{z}_2^1 \otimes (\imath_1 \oplus \jmath_{n-1}), -\delta_{\mathbb{I}_{1,2}}^2 \otimes \delta_{\mathbb{I}_{1,n}}^n\right) \end{aligned}$$

Theorem 5.22 (Recursion DCT-3).

$$\mathrm{DCT\text{-}3}_{2n}(r) = \mathrm{K}_n^{2n}\left(\bigoplus_{i=0}^{1} \mathrm{DCT\text{-}3}_n\left(\mathrm{r}_i^2(r)\right)\right)\left(\mathrm{F}_2 \otimes \mathrm{I}_n\right) B_{2n,2}^{\mathrm{DCT\text{-}3}/T,r}$$

5.4 DIT Rules for DTTs

5.4.1 DST-2, U-Basis

Definition 86 (Base Change DST-2, U-Basis).

$$B^{\mathrm{DST\text{-}2}/U}_{mn,m} = \Big(\big(\mathrm{I}_{n-1} \otimes \mathrm{S}_m^\top\big) \oplus \mathrm{I}_m \Big)\, \mathrm{L}^{mn}_n \Big(\underbrace{(\mathrm{I}_{n-1} \oplus \mathrm{I}_1) \oplus (\mathrm{J}_{n-1} \oplus \mathrm{I}_1) \oplus \cdots}_{m \text{ summands}} \Big)$$

Property 5.19 (Base Change DST-2, U-Basis).

$$B^{\mathrm{DST\text{-}2}/U}_{mn,m} = \begin{bmatrix} \mathrm{I}_{n-1} & & & & & & & & \\ & 1 & & & & & & & \\ \mathrm{J}_{n-1} & & \mathrm{I}_{n-1} & & & & & & \\ & & & 1 & & & & & \\ & & \mathrm{J}_{n-1} & & \mathrm{I}_{n-1} & & & & \\ & & & \ddots & & 1 & & & \\ & & & & \ddots & & \ddots & & \\ & & & & & \mathrm{J}_{n-1} & & \mathrm{I}_{n-1} & \\ & & & & & & & & 1 \end{bmatrix}$$

Property 5.20 (Base Change DST-2, U-Basis).

$$B^{\mathrm{DST\text{-}2}/U}_{mn,m} = \mathrm{I}_{mn} + \mathrm{H}^1_m \otimes (\mathrm{J}_{n-1} \oplus 0_{1\times 1})$$

Property 5.21 (Base Change DST-2, U-Basis).

$$B^{\mathrm{DST\text{-}2}/U}_{mn,m} = \operatorname{diag}\big(\imath^{mn\to\mathbb{C}}\big) + \operatorname{mon}\big(\mathrm{z}^{m-1}_m \otimes (\jmath_{n-1} \oplus \imath_1), \delta^m_{\mathbb{I}_{0,m-1}} \otimes \delta^n_{\mathbb{I}_{0,n-1}}\big)$$

Theorem 5.23 (Base Case DST-2).

$$\begin{aligned} \mathrm{DST\text{-}2}_2(r) &= \operatorname{diag}\big(\sin \tfrac{r\pi}{2}, \sin r\pi\big)\ \mathrm{F}_2 \\ \overline{\mathrm{DST\text{-}2}}_2(r) &= \operatorname{diag}\big(1, 2\cos \tfrac{r\pi}{2}\big)\ F_2 \end{aligned}$$

Theorem 5.24 (Recursion DST-2).

$$\begin{aligned} \mathrm{DST\text{-}2}_n &= \mathrm{DST\text{-}2}_n(1/2) \\ \mathrm{DST\text{-}2}_{mn}(r) &= B^{\mathrm{DST\text{-}2}/U}_{mn,m} \Big(\overline{\mathrm{DST\text{-}2}}_m(r) \otimes \mathrm{I}_n\Big) \Big(\bigoplus_{i=0}^{m-1} \mathrm{DST\text{-}2}_n\big(\mathrm{r}^m_i(r)\big)\Big) \mathrm{M}^{mn}_m \\ \overline{\mathrm{DST\text{-}2}}_{mn}(r) &= B^{\mathrm{DST\text{-}2}/U}_{mn,m} \Big(\overline{\mathrm{DST\text{-}2}}_m(r) \otimes \mathrm{I}_n\Big) \Big(\bigoplus_{i=0}^{m-1} \overline{\mathrm{DST\text{-}2}}_n\big(\mathrm{r}^m_i(r)\big)\Big) \mathrm{M}^{mn}_m \end{aligned}$$

5.4.2 DCT-2, U-Basis

Definition 87 (Base Change DCT-2, U-Basis).

$$B^{\mathrm{DCT\text{-}2}/U}_{mn,m} = \Big(\tfrac{1}{2}\big(2\,\mathrm{I}_1 \oplus \mathrm{I}_{m-1} - \mathrm{H}^2_m\big) \oplus \big(\mathrm{I}_{n-1} \otimes \mathrm{S}_m^\top\big)\Big)\, \mathrm{L}^{mn}_n \Big(\underbrace{(\mathrm{I}_1 \oplus \mathrm{I}_{n-1}) \oplus (\mathrm{I}_1 \oplus \mathrm{J}_{n-1}) \oplus \cdots}_{m \text{ summands}} \Big)$$

Property 5.22 (Base Change DCT-2, U-Basis)**.**

$$
B^{\mathrm{DCT\text{-}2}/U}_{mn,m} = \begin{bmatrix}
1 & & & & & & & & & & & \\
 & \mathrm{I}_{n-1} & & & & & & & & & & \\
 & & \frac{1}{2} & & & & & & & & & \\
 & \mathrm{J}_{n-1} & & \mathrm{I}_{n-1} & & & & & & & & \\
-\frac{1}{2} & & & & \frac{1}{2} & & & & & & & \\
 & \ddots & & \mathrm{J}_{n-1} & & \mathrm{I}_{n-1} & & & & & & \\
 & & \ddots & & \ddots & & \ddots & & & & & \\
 & & & \ddots & & \ddots & & \ddots & & & & \\
 & & & & -\frac{1}{2} & & \ddots & & \frac{1}{2} & & & \\
 & & & & & & & \mathrm{J}_{n-1} & & \mathrm{I}_{n-1} & & \\
 & & & & & & -\frac{1}{2} & & & & \frac{1}{2} & \\
 & & & & & & & & & \mathrm{J}_{n-1} & & \mathrm{I}_{n-1}
\end{bmatrix}
$$

Property 5.23 (Base Change DCT-2, U-Basis)**.**

$$
B^{\mathrm{DCT\text{-}2}/U}_{mn,m} = \mathrm{I}_n \oplus \big(\mathrm{I}_{m-1} \otimes (\tfrac{1}{2} \oplus \mathrm{I}_{n-1}) \big) + \mathrm{H}^1_m \otimes (0_{1\times 1} \oplus \mathrm{J}_{n-1}) - \mathrm{H}^2_m \otimes (\tfrac{1}{2} \oplus 0_{n-1\times n-1})
$$

Property 5.24 (Base Change DCT-2, U-Basis)**.**

$$
\begin{aligned}
B^{\mathrm{DCT\text{-}2}/U}_{mn,m} \quad = \quad & \mathrm{diag}\Big(\big(\delta^m_{\mathbb{I}_1} \otimes \delta^n_{\mathbb{I}_1}\big) + \tfrac{1}{2}\big(\delta^m_{\mathbb{I}_{1,m}} \otimes \delta^n_{\mathbb{I}_1}\big) + \big(\delta^m_{\mathbb{I}_m} \otimes \delta^n_{\mathbb{I}_{1,n}}\big)\Big) + \\
& + \mathrm{mon}\Big(\mathrm{z}^{m-1}_m \otimes (\imath_1 \oplus \jmath_{n-1}), \delta^m_{\mathbb{I}_{m-1}} \otimes \delta^n_{\mathbb{I}_{1,n}} \Big) + \\
& + \mathrm{mon}\Big(\mathrm{z}^{m-2}_m \otimes \imath_n, -\tfrac{1}{2}\big(\delta^m_{\mathbb{I}_{m-2}} \otimes \delta^n_{\mathbb{I}_1}\big)\Big)
\end{aligned}
$$

Theorem 5.25 (Base Case DCT-2)**.**

$$
\mathrm{DCT\text{-}3}_2(r) \quad = \quad \mathrm{diag}\,(1, \cos r\pi)\ \mathrm{F}_2
$$

Theorem 5.26 (Recursion DCT-2)**.**

$$
\begin{aligned}
\mathrm{DCT\text{-}2}_n \quad & = \quad \mathrm{DCT\text{-}2}_n(1/2) \\
\mathrm{DCT\text{-}2}_{mn}(r) \quad & = \quad B^{\mathrm{DCT\text{-}2}/U}_{mn,m} \Big(\overline{\mathrm{DCT\text{-}2}}_m(r) \otimes \mathrm{I}_n \Big) \left(\bigoplus_{i=0}^{m-1} \mathrm{DCT\text{-}2}_n\big(\mathrm{r}^m_i(r)\big) \right) \mathrm{M}^{mn}_m
\end{aligned}
$$

5.4.3 DST-4, U-Basis

Definition 88 (Base Change DST-4, U-Basis)**.**

$$
B^{\top,\mathrm{DST\text{-}4}/U}_{mn,m} = \big(\mathrm{S}^\top_m \otimes \mathrm{I}_n \big)^{\overbrace{\mathrm{I}_n \oplus \mathrm{J}_n \oplus \cdots}^{m \text{ summands}}}
$$

Property 5.25 (Base Change DST-4, U-Basis)**.**

$$
B^{\top,\mathrm{DST\text{-}4}/U}_{mn,m} = \begin{bmatrix}
\mathrm{I}_n & & & & \\
\mathrm{J}_n & \mathrm{I}_n & & & \\
 & \mathrm{J}_n & \mathrm{I}_n & & \\
 & & \ddots & \ddots & \\
 & & & \mathrm{J}_n & \mathrm{I}_n
\end{bmatrix}
$$

Property 5.26 (Base Change DST-4, U-Basis)**.**

$$
B^{\top,\mathrm{DST\text{-}4}/U}_{mn,m} = \mathrm{I}_{mn} + \mathrm{H}^1_m \otimes \mathrm{J}_n
$$

Property 5.27 (Base Change DST-4, U-Basis).

$$B^{\top,\mathrm{DST}\text{-}4/U}_{mn,m} = \mathrm{diag}\left(\imath^{mn\to\mathbb{C}}\right) + \mathrm{mon}\left(\mathrm{z}^{m-1}_m \otimes \jmath_n, \delta^m_{\mathbb{I}_{m-1}} \otimes \imath^{n\to\mathbb{C}}\right)$$

Theorem 5.27 (Base Case DST-4).

$$\mathrm{DST}\text{-}4_2(r) = \begin{bmatrix} 1 & 0 \\ 1 & 2\cos\frac{r\pi}{2} \end{bmatrix} \mathrm{F}_2 \ \mathrm{diag}\left(\sin\frac{r\pi}{4}, \cos\frac{r\pi}{4}\right)$$

Theorem 5.28 (Recursion DST-4).

$$\begin{aligned} \mathrm{DST}\text{-}4_n &= \mathrm{DST}\text{-}4_n(1/2) \\ \mathrm{DST}\text{-}4_{mn}(r) &= B^{\top,\mathrm{DST}\text{-}4/U}_{mn,m}\left(\overline{\mathrm{DST}\text{-}2}_m(r) \otimes \mathrm{I}_n\right)\left(\bigoplus_{i=0}^{m-1} \mathrm{DST}\text{-}4_n\left(\mathrm{r}^m_i(r)\right)\right)\mathrm{M}^{mn}_m \end{aligned}$$

5.4.4 DCT-4, U-Basis

Definition 89 (Base Change DCT-4, U-Basis).

$$B^{\top,\mathrm{DCT}\text{-}4/U}_{mn,m} = \left((\mathrm{I}_m - \mathrm{H}^1_m) \otimes \mathrm{I}_n\right) \underbrace{\mathrm{I}_n \oplus \mathrm{J}_n \oplus \cdots}_{m \text{ summands}}$$

Property 5.28 (Base Change DCT-4, U-Basis).

$$B^{\top\,\mathrm{DCT}\text{-}4/U}_{mn,m} = \begin{bmatrix} \mathrm{I}_n & & & & \\ -\mathrm{J}_n & \mathrm{I}_n & & & \\ & -\mathrm{J}_n & \mathrm{I}_n & & \\ & & \ddots & \ddots & \\ & & & -\mathrm{J}_n & \mathrm{I}_n \end{bmatrix}$$

Property 5.29 (Base Change DCT-4, U-Basis).

$$B^{\top,\mathrm{DCT}\text{-}4/U}_{mn,m} = \mathrm{I}_{mn} - \mathrm{H}^1_m \otimes \mathrm{J}_n$$

Property 5.30 (Base Change DCT-4, U-Basis).

$$B^{\top,\mathrm{DCT}\text{-}4/U}_{mn,m} = \mathrm{diag}\left(\imath^{mn\to\mathbb{C}}\right) + \mathrm{mon}\left(\mathrm{z}^{m-1}_m \otimes \jmath_n, \delta^m_{\mathbb{I}_{m-1}} \otimes \left(-\imath^{n\to\mathbb{C}}\right)\right)$$

Theorem 5.29 (Base Case DCT-4).

$$\mathrm{DCT}\text{-}4_2(r) = \begin{bmatrix} 1 & 0 \\ -1 & 2\cos\frac{r\pi}{2} \end{bmatrix} \mathrm{F}_2 \ \mathrm{diag}\left(\cos\frac{r\pi}{4}, \sin\frac{r\pi}{4}\right)$$

Theorem 5.30 (Recursion DCT-4).

$$\begin{aligned} \mathrm{DCT}\text{-}4_n &= \mathrm{DCT}\text{-}4_n(1/2) \\ \mathrm{DCT}\text{-}4_{mn}(r) &= B^{\top,\mathrm{DCT}\text{-}4/U}_{mn,m}\left(\overline{\mathrm{DST}\text{-}2}_m(r) \otimes \mathrm{I}_n\right)\left(\bigoplus_{i=0}^{m-1} \mathrm{DCT}\text{-}4_n\left(\mathrm{r}^m_i(r)\right)\right)\mathrm{M}^{mn}_m \end{aligned}$$

5.4.5 DCT-2, T-Basis

Definition 90 (Base Change iDCT-3, T-Basis).

$$C^{-1,\mathrm{DCT}\text{-}3/T}_{mn,m} = \left(\mathrm{I}_m \oplus (\mathrm{I}_{n-1} \otimes \mathrm{S}_m)\right) \mathrm{L}^{mn}_n \left(\underbrace{(\mathrm{I}_1 \oplus \mathrm{I}_{n-1}) \oplus (\mathrm{I}_1 \oplus \mathrm{J}_{n-1}) \oplus \cdots}_{m \text{ summands}}\right)$$

Property 5.31 (Base Change iDCT-3, T-Basis).

$$C_{mn,m}^{-1,\mathrm{DCT\text{-}3}/T} = \begin{bmatrix} 1 &&&&&&&&&&&& \\ & \mathrm{I}_{n-1} && \mathrm{J}_{n-1} &&&&&&&&& \\ && 1 &&&&&&&&&& \\ &&& \mathrm{I}_{n-1} && \mathrm{J}_{n-1} &&&&&&& \\ &&&& \ddots && \ddots &&&&&& \\ &&&&& \ddots && \ddots &&&&& \\ &&&&&& 1 && \ddots &&&& \\ &&&&&&& \mathrm{I}_{n-1} && \mathrm{J}_{n-1} &&& \\ &&&&&&&& 1 &&&& \\ &&&&&&&&& \mathrm{I}_{n-1} &&& \end{bmatrix}$$

Property 5.32 (Base Change iDCT-3, T-Basis).

$$C_{mn,m}^{-1,\mathrm{DCT\text{-}3}/T} = \mathrm{I}_{mn} + \mathrm{U}_m^1 \otimes (0_{1\times 1} \oplus \mathrm{J}_{n-1})$$

Property 5.33 (Base Change iDCT-3, T-Basis).

$$C_{mn,m}^{-1,\mathrm{DCT\text{-}3}/T} = \mathrm{diag}\left(\imath^{mn\to\mathbb{C}}\right) + \mathrm{mon}\left(\mathrm{z}_m^1 \otimes (\imath_1 \oplus \jmath_{n-1}), \delta_{\mathbb{I}_{1,m}}^m \otimes \delta_{\mathbb{I}_{1,n}}^n\right)$$

Theorem 5.31 (Base Case iDCT-3).

$$\mathrm{iDCT\text{-}3}_2(r) = \mathrm{diag}\left(1, \frac{1}{2\cos r\pi}\right) \mathrm{F}_2$$

Theorem 5.32 (Recursion DCT-2).

$$\begin{aligned} \mathrm{DCT\text{-}2}_n &= \mathrm{iDCT\text{-}3}_n(1/2) \\ \mathrm{iDCT\text{-}3}_{mn}(r) &= C_{mn,m}^{-1,\mathrm{DCT\text{-}3}/T}\left(\mathrm{iDCT\text{-}3}_m(r) \otimes \mathrm{I}_n\right)\left(\bigoplus_{i=0}^{m-1} \mathrm{iDCT\text{-}3}_n\left(\mathrm{r}_i^m(r)\right)\right) \mathrm{M}_m^{mn} \end{aligned}$$

5.4.6 DCT-2, T-Basis, Radix-2

Definition 91 (Base Change DCT-2, T-Basis, Radix-2).

$$C_{2n,2}^{\mathrm{DCT\text{-}2}/T} = \left(\mathrm{I}_2 \oplus \left(\mathrm{I}_{n-1} \otimes (\mathrm{I}_2 - \mathrm{H}_2^1)\right)\right)^{\mathrm{L}_n^{2n}}\left(\mathrm{I}_n \oplus (\mathrm{I}_1 \oplus \mathrm{J}_{n-1})\right)$$

Property 5.34 (Base Change DCT-2, T-Basis, Radix-2).

$$C_{2n,2}^{\mathrm{DCT\text{-}2}/T} = \mathrm{I}_{2n} - \mathrm{H}_2^1 \otimes (0_1 \oplus \mathrm{J}_{n-1})$$

Definition 92 (Base Change Diagonal DCT-2, T-Basis, Radix-2).

$$E_{2n,2}^{\mathrm{DCT\text{-}2}/T,r} = \mathrm{I}_n \oplus \left(\cos r\pi \ \mathrm{diag}\left(\mathrm{I}_1 \oplus 2\,\mathrm{I}_{n-1}\right)\right)$$

Property 5.35 (Base Change DCT-2, T-Basis, Radix-2).

$$B_{2n,2}^{\mathrm{DCT\text{-}2}/T,r} = \begin{bmatrix} \mathrm{I}_1 &&& \\ & \mathrm{I}_{n-1} && \\ && \mathrm{I}_1 \cos r\pi & \\ & -\mathrm{J}_{n-1} && 2\cos r\pi\, \mathrm{I}_{n-1} \end{bmatrix}$$

Property 5.36 (Base Change DCT-2, T-Basis, Radix-2).

$$\begin{aligned} B_{2n,2}^{\mathrm{DCT\text{-}2}/T,r} = {} & \mathrm{diag}\left(\left(\delta_{\mathbb{I}_1}^2 \otimes \imath^{n\to\mathbb{C}}\right) + \cos r\pi\left(\delta_{\mathbb{I}_{1,2}}^2 \otimes \delta_{\mathbb{I}_1}^n\right) + 2\cos r\pi\left(\delta_{\mathbb{I}_{1,2}}^2 \otimes \delta_{\mathbb{I}_{1,n}}^n\right)\right) \\ & + \mathrm{mon}\left(\mathrm{z}_2^1 \otimes (\imath_1 \oplus \jmath_{n-1}), -\delta_{\mathbb{I}_1}^2 \otimes \delta_{\mathbb{I}_{1,n}}^n\right) \end{aligned}$$

Theorem 5.33 (Recursion DCT-2).

$$\mathrm{DCT\text{-}3}_{2n}(r) = B_{2n,2}^{\mathrm{DCT\text{-}2}/T,r}\left(\mathrm{F}_2 \otimes \mathrm{I}_n\right)\left(\bigoplus_{i=0}^{1} \mathrm{DCT\text{-}2}_n\left(\mathrm{r}_i^2(r)\right)\right) \mathrm{M}_2^{2n}$$

6 Definitions

Definition 93 (Standard Basis)**.** Let $e_0^n, e_1^n, \ldots, e_{n-1}^n$ denote the vectors in $\mathbb{C}^{n\times 1}$ with a 1 in the component given by the subscript and 0 elsewhere. The set

$$B_n = \{e_i^n : i = 0, 1, \ldots, n-1\} \tag{1}$$

is the standard basis of $\mathbb{C}^{n\times 1}$.

6.1 Operators

Definition 94 (Matrix Sum)**.**

$$A = A_0 + A_1$$

Definition 95 (Iterative Matrix Sum)**.**

$$\sum_{i=0}^{k-1} A_i = A_0 + \ldots + A_{k-1}$$

Definition 96 (Matrix Product)**.**

$$A = A_0\, A_1$$

Definition 97 (Iterative Matrix Product)**.**

$$\prod_{i=0}^{k-1} A_i = A_0 \cdots A_{k-1}$$

Definition 98 (Matrix Direct Sum)**.**

$$A = A_0 \oplus A_1$$

Definition 99 (Iterative Matrix Direct Sum)**.**

$$\bigoplus_{i=0}^{k-1} A_i = A_0 \oplus \ldots \oplus A_{k-1}$$

Definition 100 (Row Overlapped Matrix Direct Sum)**.**

$$A = A_0 \oplus_k A_1$$

Definition 101 (Iterative Row Overlapped Matrix Direct Sum)**.**

$$\bigoplus_{i=0}^{m-1}{}_k\, A_i = A_0 \oplus_k \ldots \oplus_k A_{k-1}$$

Definition 102 (Column Overlapped Matrix Direct Sum)**.**

$$A = A_0 \oplus^k A_1$$

Definition 103 (Iterative Column Overlapped Matrix Direct Sum)**.**

$$\bigoplus_{i=0}^{m-1}{}^k\, A_i = A_0 \oplus^k \ldots \oplus^k A_{k-1}$$

Definition 104 (Iterative Vertical Stack)**.**

$$\mathop{\left[\frac{\quad}{}\right]}_{j\,=\,0}^{m\,-\,1} A_j = \left[\begin{array}{c} A_0 \\ \hline \vdots \\ \hline A_{m-1} \end{array}\right]$$

Definition 105 (Iterative Horizontal Stack)**.**

$$\left[\ \Big|\ \right]_{j=0}^{m-1} A_j = \left[\ A_0 \ \middle|\ \dots\ \middle|\ A_{m-1}\ \right]$$

Definition 106 (Matrix Tensor Product)**.**

$$A = A_0 \otimes A_1$$

Definition 107 (Matrix Row Overlapped Tensor Product)**.**

$$A = \mathrm{I}_m \otimes_k A_0 = \bigoplus_{i=0}^{m-1}{}_k\, A$$

Definition 108 (Matrix Column Overlapped Tensor Product)**.**

$$A = \mathrm{I}_m \otimes^k A_0 = \bigoplus_{i=0}^{m-1}{}^k\, A$$

Definition 109 (Iterative Matrix Tensor Product)**.**

$$\bigotimes_{i=0}^{k-1} A_i = A_0 \otimes A_1 \otimes \cdots \otimes A_{k-1}$$

Definition 110 (Matrix of Matrices)**.**

$$\begin{bmatrix} A_{00} & \cdots & A_{0n} \\ \vdots & \ddots & \vdots \\ A_{m0} & \cdots & A_{mn} \end{bmatrix}$$

Property 6.1 (Distributivity)**.**

$$\sum_{i=0}^{k-1}\Big(A_i x\Big) = \Big(\sum_{i=0}^{k-1} A_i\Big)x.$$

6.2 Generating Functions

6.2.1 Matrix Generating Functions

Definition 111 (Matrix Generating Function)**.** Matrix generating functions are of type

$$f : \begin{cases} \{0, \dots, m-1\} \times \{0, \dots, n-1\} \to \mathbb{C} \\ (i, j) \mapsto f(i, j) \end{cases} .$$

Definition 112 (Diagonal Generating Function)**.** Diagonal generating functions are of type

$$f : \begin{cases} \{0, \dots, n-1\} \to \mathbb{C} \\ i \mapsto f(i) \end{cases} .$$

Definition 113 (Diagonal Induced Matrix Generating Function)**.** The diagonal generating function

$$f : \begin{cases} \{0, \dots, n-1\} \to \mathbb{C} \\ i \mapsto f(i) \end{cases}$$

induces a matrix generation function

$$\hat{f} : \begin{cases} \{0, \ldots, n-1\} \times \{0, \ldots, n-1\} \to \mathbb{C} \\ (i, j) \mapsto \hat{f}(i, j) \end{cases}$$

with

$$\hat{f}(i, j) = \begin{cases} f(i) & \text{if } i = j \\ 0 & \text{else} \end{cases} .$$

Definition 114 (Permutation Generating Function)**.** Permutation generating functions are of type

$$\pi : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, n-1\} \\ i \mapsto \pi(i) \end{cases}$$

with the permutation

$$\pi \in \mathrm{S}_n .$$

Definition 115 (Permutation Induced Matrix Generating Function)**.** The permutation generating function

$$\pi : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, n-1\} \\ i \mapsto \pi(i) \end{cases}$$

with the permutation

$$\pi \in \mathrm{S}_n$$

induces a matrix generation function

$$\hat{\pi} : \begin{cases} \{0, \ldots, n-1\} \times \{0, \ldots, n-1\} \to \mathbb{C} \\ (i, j) \mapsto \hat{\pi}(i, j) \end{cases}$$

with

$$\hat{f}(i, j) = \begin{cases} 1 & \text{if } j = \pi(i) \\ 0 & \text{else} \end{cases} .$$

6.2.2 Index Mapping Functions

Definition 116 (Index Mapping Functions)**.** Index mapping functions are of form

$$f : \begin{cases} \{i_0, ..., i_m\} \to \{j_0, ..., j_m\} \\ i \mapsto f(i) \end{cases} .$$

Corollary 6.1 (Index Mapping Function)**.** The index mapping function

$$f : \begin{cases} \{i_0, ..., i_m\} \to \{j_0, ..., j_m\} \\ i \mapsto f(i) \end{cases}$$

induces a matrix generation function

$$\hat{f} : \begin{cases} \{0, \ldots, i_m\} \times \{0, \ldots, j_m\} \to \mathbb{C} \\ (i, j) \mapsto \hat{f}(i, j) \end{cases}$$

with

$$\hat{f}(i, j) = \begin{cases} 1 & \text{if } j = f(i) \\ 0 & \text{else} \end{cases} .$$

Corollary 6.2 (Permutation Generating Functions)**.** The permutation generating function

$$\pi : \begin{cases} \{0,\dots,n-1\} \to \{0,\dots,n-1\} \\ i \mapsto \pi(i) \end{cases}$$

with the permutation

$$\pi \in \mathrm{S}_n \,.$$

is a bijective index mapping functions.

6.3 Operations on Functions

6.3.1 Matrix Generating Functions

Definition 117 (Sum of Matrix Generating Functions)**.** The sum of the two matrix generating functions

$$f : \begin{cases} \{0,\dots,m-1\} \times \{0,\dots,n-1\} \to \mathbb{C} \\ (i,j) \mapsto f(i,j) \end{cases}$$

and

$$g : \begin{cases} \{0,\dots,m-1\} \times \{0,\dots,n-1\} \to \mathbb{C} \\ (i,j) \mapsto g(i,j) \end{cases}$$

is given by the matrix generating function

$$f+g : \begin{cases} \{0,\dots,m-1\} \times \{0,\dots,n-1\} \to \mathbb{C} \\ (i,j) \mapsto f(i,j) + g(i,j). \end{cases}$$

Definition 118 (Direct Sum of Matrix Generating Functions)**.** The direct sum of the two matrix generating functions

$$f : \begin{cases} \{0,\dots,m_0-1\} \times \{0,\dots,n_0-1\} \to \mathbb{C} \\ (i,j) \mapsto f(i,j) \end{cases}$$

and

$$g : \begin{cases} \{0,\dots,m_1-1\} \times \{0,\dots,n_1-1\} \to \mathbb{C} \\ (i,j) \mapsto g(i,j) \end{cases}$$

is given by the matrix generating function

$$f \oplus g : \begin{cases} \{0,\dots,m_0+m_1-1\} \times \{0,\dots,n_0+n_1-1\} \to \mathbb{C} \\ (i,j) \mapsto (f \oplus g)(i,j) \end{cases}$$

with

$$(f \oplus g)(i,j) = \begin{cases} f(i,j) & \text{if } (i,j) \in \{0,\dots,m_0-1\} \times \{0,\dots,n_0-1\} \\ g(i-m_0, j-n_0) & \text{if } i \in \{m_0,\dots,m_1-1\} \times \{n_0,\dots,n_1-1\} \\ 0 & \text{else} \end{cases}$$

Definition 119 (Row Overlapped Direct Sum of Matrix Generating Functions)**.** The row overlapped direct sum of the two matrix generating functions

$$f : \begin{cases} \{0,\dots,m_0-1\} \times \{0,\dots,n_0-1\} \to \mathbb{C} \\ (i,j) \mapsto f(i,j) \end{cases}$$

and

$$g : \begin{cases} \{0,\dots,m_1-1\} \times \{0,\dots,n_1-1\} \to \mathbb{C} \\ (i,j) \mapsto g(i,j) \end{cases}$$

is given by the matrix generating function

$$f \oplus_k g : \begin{cases} \{0, \dots, m_0 + m_1 - 1\} \times \{0, \dots, n_0 + n_1 - k - 1\} \to \mathbb{C} \\ (i,j) \mapsto (f \oplus_k g)(i,j) \end{cases}$$

with

$$(f \oplus_k g)(i,j) = \begin{cases} f(i,j) & \text{if } (i,j) \in \{0, \dots, m_0 - 1\} \times \{0, \dots, n_0 - 1\} \\ g(i - m_0, j - n_0 - k) & \text{if } i \in \{m_0, \dots, m_1 - 1\} \times \{n_0 - k, \dots, n_1 - k - 1\} \\ 0 & \text{else} \end{cases}$$

Definition 120 (Column Overlapped Direct Sum of Matrix Generating Functions)**.** The column overlapped direct sum of the two matrix generating functions

$$f : \begin{cases} \{0, \dots, m_0 - 1\} \times \{0, \dots, n_0 - 1\} \to \mathbb{C} \\ (i,j) \mapsto f(i,j) \end{cases}$$

and

$$g : \begin{cases} \{0, \dots, m_1 - 1\} \times \{0, \dots, n_1 - 1\} \to \mathbb{C} \\ (i,j) \mapsto g(i,j) \end{cases}$$

is given by the matrix generating function

$$f \oplus^k g : \begin{cases} \{0, \dots, m_0 + m_1 - k - 1\} \times \{0, \dots, n_0 + n_1 - 1\} \to \mathbb{C} \\ (i,j) \mapsto (f \oplus^k g)(i,j) \end{cases}$$

with

$$(f \oplus^k g)(i,j) = \begin{cases} f(i,j) & \text{if } (i,j) \in \{0, \dots, m_0 - 1\} \times \{0, \dots, n_0 - 1\} \\ g(i - m_0 - k, j - n_0) & \text{if } i \in \{m_0 - k, \dots, m_1 - k - 1\} \times \{n_0, \dots, n_1 - 1\} \\ 0 & \text{else} \end{cases}$$

Definition 121 (Multiplication of Matrix Generating Functions)**.** The multiplication of the two matrix generating functions

$$f : \begin{cases} \{0, \dots, m - 1\} \times \{0, \dots, n - 1\} \to \mathbb{C} \\ (i,j) \mapsto f(i,j) \end{cases}$$

and

$$g : \begin{cases} \{0, \dots, n - 1\} \times \{0, \dots, k - 1\} \to \mathbb{C} \\ (i,j) \mapsto g(i,j) \end{cases}$$

is given by the matrix generating function

$$fg : \begin{cases} \{0, \dots, m - 1\} \times \{0, \dots, k - 1\} \to \mathbb{C} \\ (i,j) \mapsto \sum_{r=0}^{n-1} f(i,k) g(k,j). \end{cases}$$

Definition 122 (Tensor Product of Matrix Generating Functions)**.** The tensor product of the two matrix generating functions

$$f : \begin{cases} \{0, \dots, m_0 - 1\} \times \{0, \dots, n_0 - 1\} \to \mathbb{C} \\ (i,j) \mapsto f(i,j) \end{cases}$$

and

$$g : \begin{cases} \{0, \dots, m_1 - 1\} \times \{0, \dots, n_1 - 1\} \to \mathbb{C} \\ (i,j) \mapsto g(i,j) \end{cases}$$

is given by the matrix generating function

$$f \otimes g : \begin{cases} \{0, \dots, m_0 m_1 - 1\} \times \{0, \dots, n_0 n_1 - 1\} \to \mathbb{C} \\ (i,j) \mapsto f\left(\left\lfloor \frac{i}{m_1} \right\rfloor, \left\lfloor \frac{j}{n_1} \right\rfloor\right) g\big(i \bmod m_1, j \bmod n_1\big) \end{cases} .$$

Definition 123 (Matrices of Matrix Generating Functions)**.** The matrix of the matrix generating functions

$$f_{k,l} : \begin{cases} \{0,\dots,m-1\} \times \{0,\dots,n-1\} \to \mathbb{C} \\ (i,j) \mapsto f(i,j) \end{cases} \qquad \text{with} \quad (k,l) \in \{0,\dots,K-1\} \times \{0,\dots,L-1\}$$

is given by the matrix generating function

$$\begin{bmatrix} f_{0,0} & \cdots & f_{0,L-1} \\ \vdots & \ddots & \vdots \\ f_{K-1,0} & \cdots & f_{K-1,L-1} \end{bmatrix} : \begin{cases} \{0,\dots,mK-1\} \times \{0,\dots,nL-1\} \to \mathbb{C} \\ (i,j) \mapsto f_{\lfloor \frac{i}{m} \rfloor, \lfloor \frac{j}{n} \rfloor}(i \bmod m, j \bmod n). \end{cases}$$

6.3.2 Diagonal Generating Functions

Definition 124 (Sum of Diagonal Generating Functions)**.** The sum of the two diagonal mapping functions

$$f : \begin{cases} \{0,\dots,n-1\} \to \mathbb{C} \\ i \mapsto f(i) \end{cases} \qquad \text{and} \quad g : \begin{cases} \{0,\dots,n-1\} \to \mathbb{C} \\ i \mapsto g(i) \end{cases}$$

is given by the diagonal generating function

$$f+g : \begin{cases} \{0,\dots,n-1\} \to \mathbb{C} \\ i \mapsto f(i)+g(i) \end{cases} .$$

Definition 125 (Direct Sum of Diagonal Generating Functions)**.** The direct sum of the two diagonal generating functions

$$f : \begin{cases} \{0,\dots,m-1\} \to \mathbb{C} \\ i \mapsto f(i) \end{cases} \qquad \text{and} \quad g : \begin{cases} \{0,\dots,n-1\} \to \mathbb{C} \\ i \mapsto g(i) \end{cases}$$

is given by the diagonal generation function

$$f \oplus g : \begin{cases} \{0,\dots,m+n-1\} \to \mathbb{C} \\ i \mapsto (f \oplus g)(i) \end{cases}$$

with

$$(f \oplus g)(i) = \begin{cases} f(i) & \text{if } i \in \{0,\dots,m-1\} \\ g(i-m) & \text{if } i \in \{m,\dots,n-1\} \end{cases} .$$

Definition 126 (Product of Diagonal Generating Functions)**.** The product of the two diagonal generating functions

$$f : \begin{cases} \{0,\dots,n-1\} \to \mathbb{C} \\ i \mapsto f(i) \end{cases} \qquad \text{and} \quad g : \begin{cases} \{0,\dots,n-1\} \to \mathbb{C} \\ i \mapsto g(i) \end{cases}$$

is given by the diagonal generation function

$$fg : \begin{cases} \{0,\dots,n-1\} \to \mathbb{C} \\ i \mapsto f(i)g(i) \end{cases} .$$

Definition 127 (Tensor Product of Diagonal Generating Functions)**.** The tensor product of the two diagonal generating functions

$$f : \begin{cases} \{0,\dots,m-1\} \to \mathbb{C} \\ i \mapsto f(i) \end{cases} \qquad \text{and} \quad g : \begin{cases} \{0,\dots,n-1\} \to \mathbb{C} \\ i \mapsto g(i) \end{cases}$$

is given by the diagonal generation function

$$fg : \begin{cases} \{0,\dots,mn-1\} \to \mathbb{C} \\ (i,j) \mapsto f\left(\lfloor \frac{i}{n} \rfloor\right) g\big(i \bmod n\big) \end{cases} .$$

6.3.3 Permutation Generating Functions

Definition 128 (Direct Sum of Permutation Generating Functions)**.** The direct sum of the two permutation generating functions

$$\pi : \begin{cases} \{0, \ldots, m-1\} \to \{0, \ldots, m-1\} \\ i \mapsto \pi(i) \end{cases}$$

with $\pi \in \mathrm{S}_m$ and

$$\sigma : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, n-1\} \\ i \mapsto \sigma(i) \end{cases}$$

with $\sigma \in \mathrm{S}_n$ is given by the permutation generation function

$$\pi \oplus \sigma : \begin{cases} \{0, \ldots, m+n-1\} \to \{0, \ldots, m+n-1\} \\ i \mapsto (\pi \oplus \sigma)(i) \end{cases}$$

with

$$(\pi \oplus \sigma)(i) = \begin{cases} \pi(i) & \text{if } i \in \{0, \ldots, m-1\} \\ \sigma(i-m)+m & \text{if } i \in \{m, \ldots, n-1\} \end{cases}$$

and $\pi \oplus \sigma \in \mathrm{S}_{m+n}$.

Definition 129 (Product of Permutation Generating Functions)**.** The product of the two permutation generating functions

$$\pi : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, n-1\} \\ i \mapsto \pi(i) \end{cases}$$

with $\pi \in \mathrm{S}_n$ and

$$\sigma : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, n-1\} \\ i \mapsto \sigma(i) \end{cases}$$

with $\sigma \in \mathrm{S}_n$ is given by the permutation generation function

$$\pi\sigma : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, n-1\} \\ i \mapsto (\pi \circ \sigma)(i) \end{cases}$$

with $\pi\sigma \in \mathrm{S}_n$.

Definition 130 (Tensor Product of Permutation Generating Functions)**.** The tensor product of the two permutation generating functions

$$\pi : \begin{cases} \{0, \ldots, m-1\} \to \{0, \ldots, m-1\} \\ i \mapsto \pi(i) \end{cases}$$

with $\pi \in \mathrm{S}_m$ and

$$\sigma : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, n-1\} \\ i \mapsto \sigma(i) \end{cases}$$

with $\sigma \in \mathrm{S}_n$ is given by the permutation generation function

$$\pi \otimes \sigma : \begin{cases} \{0, \ldots, mn-1\} \to \{0, \ldots, mn-1\} \\ i \mapsto n\pi\left(\left\lfloor \frac{i}{n} \right\rfloor\right) + \sigma\big(i \bmod n\big) \end{cases}$$

with $\pi \otimes \sigma \in \mathrm{S}_{mn}$.

Definition 131 (Inversion of Permutation Generating Functions)**.** The inverse of a permutation generating functions

$$\pi : \begin{cases} \{0,\ldots,n-1\} \to \{0,\ldots,n-1\} \\ i \mapsto \pi(i) \end{cases} \qquad \text{with} \quad \pi \in \mathrm{S}_n$$

is given by the permutation generation function

$$\pi^{-1} : \begin{cases} \{0,\ldots,n-1\} \to \{0,\ldots,n-1\} \\ i \mapsto j & \text{with } j = \pi(i) \end{cases}.$$

Lemma 6.1. For two permutation generating functions

$$\pi \in \mathrm{S}_m \quad \text{and} \quad w \in \mathrm{S}_n$$

it holds that

$$\begin{aligned} (\pi \circ w)^{-1} &= w^{-1} \circ \pi^{-1} \\ (\pi \oplus w)^{-1} &= \pi^{-1} \oplus w^{-1} \\ (\pi \otimes w)^{-1} &= \pi^{-1} \otimes w^{-1}. \end{aligned}$$

6.3.4 Index Mapping Functions

Definition 132 (Concatenation of Index Mapping Functions)**.** The concatenation of the two index mapping functions

$$f : \begin{cases} I \to J \\ i \mapsto f(i) \end{cases} \qquad \text{and} \quad g : \begin{cases} J \to K \\ i \mapsto g(i) \end{cases}$$

is given by the index mapping function

$$g \circ f : \begin{cases} I \to K \\ i \mapsto g(f(i)). \end{cases}$$

Definition 133 (Restriction of Index Mapping Functions)**.** For an index mapping function f with

$$f : \begin{cases} I \to J \\ i \mapsto f(i) \end{cases},$$

the restriction of f to $I_1 \subseteq I$ is defined by

$$f|_{I_1} : \begin{cases} I_1 \to J \\ i \mapsto f(i) & \text{with } i \in I_1 \end{cases}.$$

Definition 134 (Fusion of Index Mapping Functions)**.** The fusion of two index mapping functions

$$f : \begin{cases} I \to J \\ i \mapsto f(i) \end{cases} \qquad \text{and} \quad g : \begin{cases} K \to L \\ i \mapsto g(i) \end{cases} \qquad \text{with} \quad I \cap K = \varnothing$$

is given by the generating function

$$f \cup g : \begin{cases} I \cup K \to J \cup L \\ i \mapsto (f \cup g)(i) \end{cases} \qquad \text{with} \quad (f \cup g)(i) = \begin{cases} f(i) & \text{if } i \in I \\ g(i) & \text{if } i \in K \end{cases}.$$

Definition 135 (Splitting of Index Mapping Functions)**.** An index mapping function

$$f : \begin{cases} I \to J \\ i \mapsto f(i) \end{cases}$$

is split into the family of functions

$$\{f_j\}_{j=0,\ldots,k-1} \quad , \quad f_j : \begin{cases} I_j \to J \\ i \mapsto f_j(i) \end{cases}$$

with

$$f = \bigcup_{j=0}^{k-1} f_j$$

by partitioning the domain of f into the domains of f_j,

$$I = \bigcup_{j=0}^{k-1} I_j \quad , \quad I_k \cap I_l = \varnothing \text{ for } k \neq l,$$

and defining

$$f_j := f|_{I_j}.$$

Definition 136 (Pseudo Inversion of Index Mapping Functions)**.** For an *injective* index mapping function f with

$$f : \begin{cases} I \to J \\ i \mapsto f(i) \end{cases} \quad ,$$

the pseudo inverse f^{-1} is defined by

$$f^{-1} : \begin{cases} f(I) \to I \\ i \mapsto j \quad \text{with } f(j) = i \end{cases}$$

6.4 Index Mapping Functions of Special Type

6.4.1 Interval Mapping Functions

Definition 137 (Interval Mapping Function)**.** A index mapping function of form

$$f : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, N-1\} \\ i \mapsto f(i) \end{cases}$$

is called interval mapping function.

Definition 138 (Identity Interval Mapping Function)**.** The identity interval mapping function is given by

$$\imath_n : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, n-1\} \\ i \mapsto i \end{cases} \quad .$$

Definition 139 (Basis Interval Mapping Function)**.** Basis-n interval mapping functions are given by

$$(j)_n : \begin{cases} \{0\} \to \{0, \ldots, n-1\} \\ i \mapsto j \end{cases} \quad \text{with} \quad 0 \leq j < n.$$

If the value of m is clear from the context, the shortcut

$$j := (j)_m \quad \text{with} \quad 0 \leq j < n.$$

is used.

Property 6.2 (Concatenation of Interval Mapping Functions)**.** The concatenation of the two interval mapping functions

$$f : \begin{cases} \{0, \dots, n-1\} \to \{0, \dots, N_1 - 1\} \\ i \mapsto f(i) \end{cases}$$

and

$$g : \begin{cases} \{0, \dots, N_1-1\} \to \{0, \dots, N - 1\} \\ i \mapsto g(i) \end{cases}$$

is the interval mapping function

$$g \circ f : \begin{cases} \{0, \dots, n-1\} \to \{0, \dots, N - 1\} \\ i \mapsto g\big(f(i)\big) \end{cases}$$

Definition 140 (Direct Sum of Interval Mapping Functions)**.** The direct sum of the two interval mapping functions

$$f : \begin{cases} \{0, \dots, m-1\} \to \{0, \dots, M - 1\} \\ i \mapsto f(i) \end{cases}$$

and

$$g : \begin{cases} \{0, \dots, n-1\} \to \{0, \dots, N - 1\} \\ i \mapsto g(i) \end{cases}$$

is given by the interval mapping function

$$f \oplus g : \begin{cases} \{0, \dots, m+n-1\} \to \{0, \dots, M + N - 1\} \\ i \mapsto \begin{cases} f(i) & \text{if } i \in \{0, \dots, m-1\} \\ g(i-m) + m & \text{if } i \in \{m, \dots, n-1\} \end{cases} \end{cases} .$$

Definition 141 (Tensor Product of Interval Mapping Functions)**.** The tensor product of the two interval mapping functions

$$f : \begin{cases} \{0, \dots, m-1\} \to \{0, \dots, M - 1\} \\ i \mapsto f(i) \end{cases}$$

and

$$g : \begin{cases} \{0, \dots, n-1\} \to \{0, \dots, N - 1\} \\ i \mapsto g(i) \end{cases}$$

is given by the interval mapping function

$$f \otimes g : \begin{cases} \{0, \dots, mn-1\} \to \{0, \dots, MN - 1\} \\ i \mapsto nf\left(\left\lfloor \frac{i}{n} \right\rfloor\right) + g\big(i \bmod n\big) \end{cases} .$$

Definition 142 (Overlapped Tensor Product of Interval Mapping Functions)**.** The overlapped tensor product of the two interval mapping functions

$$f : \begin{cases} \{0, \dots, m-1\} \to \{0, \dots, M - 1\} \\ i \mapsto f(i) \end{cases}$$

and

$$g : \begin{cases} \{0, \dots, n-1\} \to \{0, \dots, N - 1\} \\ i \mapsto g(i) \end{cases}$$

is given by the interval mapping function

$$f \otimes_k g : \begin{cases} \{0, \dots, mn-1\} \to \{0, \dots, MN - 1\} \\ i \mapsto (n-k)f\left(\left\lfloor \frac{i}{n} \right\rfloor\right) + g\big(i \bmod n\big) \end{cases} .$$

Definition 143 (Stacking of Interval Mapping Functions)**.** The stack of the two interval mapping functions

$$f : \begin{cases} \{0, \dots, n_0 - 1\} \to \{0, \dots, N - 1\} \\ i \mapsto f(i) \end{cases}$$

and

$$g : \begin{cases} \{0, \dots, n_1 - 1\} \to \{0, \dots, N - 1\} \\ i \mapsto g(i) \end{cases}$$

is given by the interval mapping function

$$\begin{bmatrix} f \\ g \end{bmatrix} : \begin{cases} \{0, \dots, n_0 + n_1 - 1\} \to \{0, \dots, N - 1\} \\ i \mapsto \left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right](i) \end{cases}$$

with

$$\begin{bmatrix} f \\ g \end{bmatrix}(i) = \begin{cases} f(i) & \text{if } (i, j) \in \{0, \dots, n_0 - 1\} \\ g(i - n_0) & \text{if } i \in \{n_0, \dots, n_1 - 1\}. \end{cases}$$

Theorem 6.1 (Decomposition into Interval Mapping Functions)**.** Any index mapping function

$$f : \begin{cases} \{i_0, ..., i_{m-1}\} \to \{j_0, ..., j_{m-1}\} \\ i \mapsto f(i) \end{cases} \quad .$$

can be decomposed into a concatenation of two interval mapping functions,

$$f = r \circ w^{-1},$$

with

$$r : \begin{cases} \{0, \dots, m - 1\} \to \{0, ..., N - 1\} \\ i \mapsto r(i) \end{cases} \qquad \text{with} \quad N - 1 \geq j_{m-1}$$

and

$$w : \begin{cases} \{0, \dots, m - 1\} \to \{0, ..., N - 1\} \\ i \mapsto w(i) \end{cases} \qquad \text{with} \quad N - 1 \geq i_{m-1}.$$

Definition 144 (Locality)**.** The locality of a family of interval mapping functions

$$\{f_j\}_{j=0,\dots,m-1} \quad \text{with} \quad f_j : \begin{cases} \{0, \dots, n_j - 1\} \to \{0, \dots, N - 1\} \\ i \mapsto f_j(i) \end{cases}$$

is defined by

$$\Lambda\left(\{f_j\}_{j=0,\dots,m-1}\right) := \left\{f_0\big(\{0, \dots, n_j - 1\}\big), \dots, f_{m-1}\big(\{0, \dots, n_j - 1\}\big)\right\}.$$

Lemma 6.2. From

$$|M| \neq |N|$$

follows that

$$\Lambda\Big(\{r_j\}_{j\in M}\Big) \neq \Lambda\Big(\{w_j\}_{j\in N}\Big).$$

6.4.2 Additive Separable Functions

Definition 145 (Additive **k**-Separability)**.** A family of interval mapping functions

$$\{f_j\}_{j=0,\dots,m-1}$$

is additive k-separable if the family $\{f_j\}$ has a common closed form

$$f_j : \begin{cases} \{0,\dots,n_{\iota(j)}-1\} \to \{0,\dots,N-1\} \\ i \mapsto b(j) + s_{\iota(j)}(i) \end{cases} \qquad \text{for} \quad 0 \le j < m$$

with the "base function"

$$b : \begin{cases} \{0,\dots,m-1\} \to \{0,\dots,N-1\} \\ j \mapsto b(j) \end{cases}$$

and the family of "stride functions"

$$\{s_l\}_{l=0,\dots,k-1} \quad \text{with} \quad s_l : \begin{cases} \{0,\dots,n_l-1\} \to \{0,\dots,N-1\} \\ i \mapsto s'(i,l) \end{cases}$$

and the "stride instantiation" function

$$\iota : \begin{cases} \{0,\dots,m-1\} \to \{0,\dots,k-1\} \\ j \mapsto \iota(j). \end{cases}$$

Definition 146 (Additive Separability)**.** A family of interval mapping functions

$$\{f_j\}_{j=0,\dots,m-1}$$

is additive separable if it is additive 1-separable, i. e., the family $\{f_j\}$ has a stride function $s(i)$ that is independent of j:

$$f_j : \begin{cases} \{0,\dots,n-1\} \to \{0,\dots,N-1\} \\ i \mapsto b(j) + s(i) \end{cases} \qquad \text{for} \quad 0 \le j < m.$$

Lemma 6.3. For two families of additive separable functions

$$f_j : \begin{cases} \{0,\dots,n-1\} \to \{0,\dots,N-1\} \\ i \mapsto b(j) + s(i) \end{cases} \qquad \text{with } j = 0,\dots,m-1$$

and

$$g_j : \begin{cases} \{0,\dots,n-1\} \to \{0,\dots,N-1\} \\ i \mapsto c(j) + t(i) \end{cases} \qquad \text{with } j = 0,\dots,m-1$$

the condition

$$\Lambda\Big(\{f_j\}_{j=0,\dots,m-1}\Big) = \Lambda\Big(\{g_j\}_{j=0,\dots,m-1}\Big)$$

is equivalent to the two conditions

$$\Lambda\big(b(j)\big) = \Lambda\big(c(j)\big) \quad \text{and} \quad \Lambda\big(s(j)\big) = \Lambda\big(t(j)\big) \quad \forall j = 0,\dots,m-1.$$

Lemma 6.4. For two families of additive separable functions

$$f_j : \begin{cases} \{0,\dots,n-1\} \to \{0,\dots,N-1\} \\ i \mapsto b(j) + s(i) \end{cases} \qquad \text{with } j = 0,\dots,m-1$$

and

$$g_j : \begin{cases} \{0,\dots,n-1\} \to \{0,\dots,N-1\} \\ i \mapsto c(j) + t(i) \end{cases} \qquad \text{with } j = 0,\dots,m-1$$

the condition

$$\Lambda\big(f_j\big) = \Lambda\big(g_j\big) \quad \forall j = 0, \ldots, m-1.$$

is equivalent to the two conditions

$$b(j) = c(j) \quad \text{and} \quad \Lambda\big(s(j)\big) = \Lambda\big(t(j)\big) \quad \forall j = 0, \ldots, m-1.$$

Lemma 6.5. For two families of additive separable functions

$$f_j : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, N-1\} \\ i \mapsto b(j) + s(i) \end{cases} \qquad \text{with } j = 0, \ldots, m-1$$

and

$$g_j : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, N-1\} \\ i \mapsto c(j) + t(i) \end{cases} \qquad \text{with } j = 0, \ldots, m-1$$

and a permutation

$$\pi : \begin{cases} \{0, \ldots, m-1\} \to \{0, \ldots m-1\} \\ i \mapsto \pi(i) \end{cases} \quad , \quad \pi \in \mathrm{S}_m,$$

the condition

$$f_j = g_{\pi(j)} \quad \forall j = 0, \ldots, m-1.$$

is equivalent to the two conditions

$$b(j) = c(\pi(j)) \quad \text{and} \quad s(j) = t(j) \quad \forall j = 0, \ldots, m-1$$

and

$$\Lambda\big(b(j)\big) = \Lambda\big(c(j)\big) \quad \text{and} \quad s(j) = t(j) \quad \forall j = 0, \ldots, m-1.$$

Lemma 6.6. For two families of additive separable functions

$$f_j : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, N-1\} \\ i \mapsto b(j) + s(i) \end{cases} \qquad \text{with } j = 0, \ldots, m-1$$

and

$$g_j : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, N-1\} \\ i \mapsto c(j) + t(i) \end{cases} \qquad \text{with } j = 0, \ldots, m-1$$

the condition

$$f_j = g_j \quad \forall j = 0, \ldots, m-1.$$

is equivalent to the two conditions

$$b(j) = c(j) \quad \text{and} \quad s(j) = t(j) \quad \forall j = 0, \ldots, m-1.$$

6.4.3 Stride Functions

Definition 147 (Stride Function)**.** A interval mapping function of form

$$f : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, N-1\} \\ i \mapsto b + s\, i \end{cases}$$

is called stride function.

Property 6.3 (Concatenation of Stride Functions)**.** The concatenation of the two stride functions

$$f : \begin{cases} \{0,\dots,n-1\} \to \{0,\dots,N_1-1\} \\ i \mapsto b_1 + s_1\, i \end{cases}$$

and

$$g : \begin{cases} \{0,\dots,N_1-1\} \to \{0,\dots,N-1\} \\ i \mapsto b_2 + s_2\, i \end{cases}$$

is the stride function

$$g \circ f : \begin{cases} \{0,\dots,n-1\} \to \{0,\dots,N-1\} \\ i \mapsto (b_1 s_2 + b_2) + (s_1 s_2)\, i \end{cases}$$

Lemma 6.7 (Inversion of Stride Functions)**.** The family of stride functions

$$\{f_j\}_{j=0,\dots,m-1} \quad \text{with} \quad f_j : \begin{cases} \{0,\dots,n-1\} \to \{0,\dots,mn-1\} \\ i \mapsto j + im \end{cases}$$

has the family of pseudo inverses

$$\left\{f_j^{-1}\right\}_{j=0,\dots,m-1} \quad \text{with} \quad f_j^{-1} : \begin{cases} \{j, j+m, \dots, j+(n-1)m\} \to \{0,\dots,n-1\} \\ i \mapsto \left\lfloor \frac{i}{m} \right\rfloor \end{cases} .$$

Definition 148 (Unit Stride Function)**.** A stride function of form

$$f : \begin{cases} \{0,\dots,n-1\} \to \{0,\dots,N-1\} \\ i \mapsto b + i \end{cases}$$

is called unit stride function.

Property 6.4 (Concatenation of Unit Stride Functions)**.** The concatenation of the two unit stride functions

$$f : \begin{cases} \{0,\dots,n-1\} \to \{0,\dots,N_1-1\} \\ i \mapsto b_1 + i \end{cases}$$

and

$$g : \begin{cases} \{0,\dots,N_1-1\} \to \{0,\dots,N-1\} \\ i \mapsto b_2 + i \end{cases}$$

is the unit stride function

$$g \circ f : \begin{cases} \{0,\dots,n-1\} \to \{0,\dots,N-1\} \\ i \mapsto (b_1 + b_2) + i \end{cases}$$

Definition 149 (Additive Linear Separability)**.** A family of interval mapping functions

$$\{f_j\}_{j=0,\dots,m-1}$$

is additive linear separable if the family $\{f_j\}$ it is additive separable with a stride "base function"

$$b : \begin{cases} \{0,\dots,m-1\} \to \{0,\dots,N-1\} \\ j \mapsto uj + v \end{cases} .$$

Lemma 6.8 (Inversion of Unit Stride Functions)**.** The family of unit stride functions

$$\{f_j\}_{j=0,\dots,m-1} \quad \text{with} \quad f_j : \begin{cases} \{0,\dots,n-1\} \to \{0,\dots,mn-1\} \\ i \mapsto jn + i \end{cases}$$

has the family of pseudo inverses

$$\left\{f_j^{-1}\right\}_{j=0,\dots,m-1} \quad \text{with} \quad f_j^{-1} : \begin{cases} \{nj,\dots,n(j+1)-1\} \to \{0,\dots,n-1\} \\ i \mapsto i \bmod n \end{cases} .$$

6.4.4 Simplification Rules for Special Functions

In the following, $a, b, i, j, n \in \mathbb{N}_0$.

Lemma 6.9. For $0 \leq i < n$ it holds that

$$\left\lfloor j + \frac{i}{n} \right\rfloor = j.$$

Lemma 6.10. For $0 \leq i < n$ and $an \leq b$ it holds that

$$\left\lfloor \frac{j}{a} + \frac{i}{b} \right\rfloor = \left\lfloor \frac{j}{a} \right\rfloor .$$

Lemma 6.11. For any a and b it holds that

$$(a + b) \bmod n = (a \bmod n) + (b \bmod n).$$

Lemma 6.12. For $0 \leq i < n$ it holds that

$$(jn + i) \bmod n = i$$

Lemma 6.13. For any a and b it holds that

$$(ab) \bmod (an) = a(b \bmod n).$$

Lemma 6.14. For any a it holds that

$$\left\lfloor j + \frac{i}{a} \right\rfloor = j + \left\lfloor \frac{i}{a} \right\rfloor .$$

Lemma 6.15. For any a it holds that

$$an \bmod n = 0.$$

Lemma 6.16. For any a,b, and c it holds that

$$\frac{ai + bj}{c} = i\frac{a}{c} + j\frac{b}{c}$$

Lemma 6.17. The inverse of the function

$$f : \begin{cases} \{0, \ldots, mn - 1\} \to \{0, \ldots, mn - 1\} \\ i \mapsto \left\lfloor \frac{i}{n} \right\rfloor + m(i \bmod n) \end{cases}$$

is

$$f^{-1} : \begin{cases} \{0, \ldots, mn - 1\} \to \{0, \ldots, mn - 1\} \\ i \mapsto \left\lfloor \frac{i}{m} \right\rfloor + n(i \bmod m) \end{cases}$$

6.5 Parametrized Matrices

6.5.1 General Matrix

Definition 150 (General Matrix)**.** The generating function

$$f : \begin{cases} \{0, \ldots, m - 1\} \times \{0, \ldots, n - 1\} \to \mathbb{C} \\ (i, j) \mapsto f(i, j) \end{cases} .$$

generates the matrix

$$\operatorname{matrix}(f) = \Big(f(i, j)\Big)_{\substack{i = 0, \ldots, m-1 \\ j = 0, \ldots, n-1}} \in \mathbb{C}^{m \times n}.$$

Theorem 6.2 (Compatibility)**.** Matrix generating function operations and matrix operators are compatible.

$$\begin{array}{rcl}
\operatorname{matrix}(f) + \operatorname{matrix}(g) & = & \operatorname{matrix}(f+g) \\
\operatorname{matrix}(f) \oplus \operatorname{matrix}(g) & = & \operatorname{matrix}(f \oplus g) \\
\operatorname{matrix}(f) \oplus_k \operatorname{matrix}(g) & = & \operatorname{matrix}(f \oplus_k g) \\
\operatorname{matrix}(f) \oplus^k \operatorname{matrix}(g) & = & \operatorname{matrix}(f \oplus^k g) \\
\operatorname{matrix}(f)\ \operatorname{matrix}(g) & = & \operatorname{matrix}(f\, g) \\
\operatorname{matrix}(f) \otimes \operatorname{matrix}(g) & = & \operatorname{matrix}(f \otimes g) \\
\begin{bmatrix} \operatorname{matrix}(f_{00}) & \cdots & \operatorname{matrix}(f_{0n}) \\ \vdots & \ddots & \vdots \\ \operatorname{matrix}(f_{m0}) & \cdots & \operatorname{matrix}(f_{mn}) \end{bmatrix} & = & \operatorname{matrix}\left(\begin{bmatrix} f_{00} & \cdots & f_{0n} \\ \vdots & \ddots & \vdots \\ f_{m0} & \cdots & f_{mn} \end{bmatrix}\right)
\end{array}$$

6.5.2 Diagonal Matrices

Definition 151 (Diagonal Matrix)**.** The generating function

$$f : \begin{cases} \{0,\dots,n-1\} \to \mathbb{C} \\ i \mapsto f(i) \end{cases} .$$

generates the diagonal matrix

$$\operatorname{diag}(f) = \operatorname{diag}\left(f(0),\dots,f(n-1)\right) \in \mathbb{C}^{n\times n}.$$

Thus,

$$\operatorname{diag}(f) = \operatorname{matrix}(\hat{f})$$

with $\hat{f}$ being the induced matrix generating function.

Theorem 6.3 (Compatibility)**.** Diagonal matrix generating function operations and matrix operators are compatible.

$$\begin{array}{rcl}
\operatorname{diag}(f) + \operatorname{diag}(g) & = & \operatorname{diag}(f+g) \\
\operatorname{diag}(f) \oplus \operatorname{diag}(g) & = & \operatorname{diag}(f \oplus g) \\
\operatorname{diag}(f)\ \operatorname{diag}(g) & = & \operatorname{diag}(fg) \\
\operatorname{diag}(f) \otimes \operatorname{diag}(g) & = & \operatorname{diag}(f \otimes g)
\end{array}$$

6.5.3 Permutation Matrices

Definition 152 (Permutation Matrix)**.** The generating function

$$\pi : \begin{cases} \{0,\dots,n-1\} \to \{0,\dots,n-1\} \\ i \mapsto \pi(i) \end{cases}$$

with the permutation

$$\pi \in \mathrm{S}_n$$

generates the permutation

$$\operatorname{perm}(\pi) = \operatorname{matrix}(\hat{\pi}).$$

with $\hat{\pi}$ being the matrix generating function induced by π.

Corollary 6.3 (Parametrized Permutation)**.** For

$$x = (x_0, \dots, x_{n-1})^\top \quad \text{and} \quad y = (y_0, \dots, y_{n-1})^\top \in \mathbb{C}^{n\times 1},$$

the multiplication

$$y = \mathrm{perm}(\pi)\, x$$

produces the vector y with the property

$$y_i = x_{\pi(i)}.$$

Theorem 6.4 (Compatibility)**.** Permutation matrix generating function operations and matrix operators are compatible.

$$\begin{aligned} \mathrm{perm}(\pi) \oplus \mathrm{perm}(\sigma) &= \mathrm{perm}(\pi \oplus \sigma) \\ \mathrm{perm}(\pi)\ \mathrm{perm}(\sigma) &= \mathrm{perm}(\pi\sigma) \\ \mathrm{perm}(\pi) \otimes \mathrm{perm}(\sigma) &= \mathrm{perm}(\pi \otimes \sigma) \end{aligned}$$

6.5.4 Matrices Induced by Index Mapping Functions

Definition 153 (Splitting of Induced Matrices)**.** A matrix

$$\mathrm{matrix}\left(\hat{f}\right)$$

induced by an index mapping function

$$f : \begin{cases} I \to J \\ i \mapsto f(i) \end{cases}$$

with

$$f = \bigcup_{j=0}^{k-1} f_j \quad , \quad f_j : \begin{cases} I_j \to J_j \\ i \mapsto f_j(i) \end{cases}$$

can be split into a sum of induced matrices.

$$\mathrm{matrix}\left(\widehat{\bigcup_{j=0}^{k-1} f_j}\right) = \sum_{j=0}^{k-1} \mathrm{matrix}\left(\hat{f}_j\right).$$

6.5.5 Gather Matrices

Definition 154 (Gather Matrix)**.** The gather matrix

$$\mathrm{G}_f^{N,n} := \mathrm{matrix}\left(\hat{f}\right)$$

parametrized by the interval mapping function

$$f : \begin{cases} \{0, \dots, n-1\} \to \{0, \dots, N-1\} \\ i \mapsto f(i) \end{cases}$$

is defined via the matrix generating function $\hat{f}$ induced by f. A shorthand notation used is

$$\mathrm{G}_{i \mapsto f(i)}^{N,n} := \mathrm{G}_f^{N,n} \quad \text{with} \quad f : \begin{cases} \{0, \dots, n-1\} \to \{0, \dots, N-1\} \\ i \mapsto f(i) \end{cases} ,$$

omitting the domain and range of f as these values are encoded in the parameters N and n of $\mathrm{G}_f^{N,n}$.

Property 6.5 (Gather Matrix)**.** Gather matrices are stacks of transposed basis vectors:

$$\mathrm{G}_f^{N,n} := \left(\begin{array}{c} {e_{f(0)}^{N}}^\top \\ \hline {e_{f(1)}^{N}}^\top \\ \hline \vdots \\ \hline {e_{f(n-1)}^{N}}^\top \end{array}\right),$$

where $e_i^N \in \mathbb{C}^{N\times 1}$ is a vector of the standard basis (1).

Corollary 6.4 (Application of Gather Matrices)**.** For

$$x = (x_0, \dots, x_{N-1})^\top \in \mathbb{C}^{N\times 1} \quad \text{and} \quad y = (y_0, \dots, y_{n-1})^\top \in \mathbb{C}^{n\times 1},$$

the multiplication

$$y = \mathrm{G}_f^{N,n}\, x$$

produces the vector y with the property

$$y_i = x_{f(i)}.$$

Corollary 6.5 (Gather Matrices and Standard Bases)**.**

$$\mathrm{G}_f^{N,n}\, e_j^N = \begin{cases} e_i^n & \text{if } j = f(i) \\ 0^n & \text{else} \end{cases}$$

Example 6.1 (Gather Matrix)**.** For $x \in \mathbb{C}^{8\times 1}$, $y \in \mathbb{C}^{4\times 1}$, $y := \mathrm{G}_{i\mapsto 2i}^{8,4}\, x$ is given by

$$y \quad := \quad \mathrm{G}_{i\mapsto 2i}^{8,4}\, x \quad = \quad \begin{pmatrix} 1 & . & . & . & . & . & . & . \\ . & . & 1 & . & . & . & . & . \\ . & . & . & . & 1 & . & . & . \\ . & . & . & . & . & . & 1 & . \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} x_0 \\ x_2 \\ x_4 \\ x_6 \end{pmatrix}$$

with the zeros represented by dots.

6.5.6 Scatter Matrices

Definition 155 (Scatter Matrix)**.** The scatter matrix

$$\mathrm{S}_f^{N,n} := \left(\mathrm{G}_f^{N,n}\right)^\top$$

parametrized by the interval mapping function

$$f : \begin{cases} \{0, \dots, n-1\} \to \{0, \dots, N-1\} \\ i \mapsto f(i) \end{cases}$$

is defined as the transpose of the corresponding gather matrix $\mathrm{G}_f^{N,n}$. A shorthand notation used is

$$\mathrm{S}_{i\mapsto f(i)}^{N,n} := \mathrm{S}_f^{N,n} \quad \text{with} \quad f : \begin{cases} \{0, \dots, n-1\} \to \{0, \dots, N-1\} \\ i \mapsto f(i) \end{cases} ,$$

omitting the domain and range of f as these values are encoded in the parameters N and n of $\mathrm{S}_f^{N,n}$.

Property 6.6 (Scatter Matrix)**.** Scatter matrices are rows of basis vectors.

$$\mathrm{S}_f^{N,n} := \left(e_{f(0)}^N \mid e_{f(1)}^N \mid \cdots \mid e_{f(n-1)}^N\right),$$

where $e_i^N \in \mathbb{C}^{N\times 1}$ is a vector of the standard basis (1).

Corollary 6.6 (Application of Scatter Matrices)**.** For

$$x = (x_0, \dots, x_{n-1})^\top \in \mathbb{C}^{n\times 1} \quad \text{and} \quad y = (y_0, \dots, y_{N-1})^\top \in \mathbb{C}^{N\times 1},$$

the multiplication

$$y = \mathrm{S}_f^{N,n}\ x$$

produces the vector y with the property

$$y_j = \begin{cases} x_i & \text{if } j = f(i) \\ 0 & \text{else} \end{cases}$$

Corollary 6.7 (Scatter Matrices and Standard Bases)**.**

$$\mathrm{S}_f^{N,n}\ e_i^n = e_{f(i)}^N$$

Example 6.2 (Scatter Matrix)**.** For $x \in \mathbb{C}^{4\times 1}$, $y \in \mathbb{C}^{8\times 1}$, $y := \mathrm{S}_{i\mapsto i+4}^{8,4}\ x$ is given by

$$y \quad := \quad \mathrm{S}_{i\mapsto i+4}^{8,4}\ x \quad = \quad \begin{pmatrix} . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ 1 & . & . & . \\ . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_8 \end{pmatrix} \quad = \quad \begin{pmatrix} . \\ . \\ . \\ . \\ x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

with the zeros represented by dots.

6.6 Algebraic Properties of Gather and Scatter Matrices

Property 6.7 (Trivial Gather Matrix)**.**

$$\mathrm{G}_{\mathrm{id}}^{N,N} = \mathrm{I}_N$$

Property 6.8 (Trivial Scatter Matrix)**.**

$$\mathrm{S}_{\mathrm{id}}^{N,N} = \mathrm{I}_N$$

Transposition of gather matrices yields scatter matrices.

Property 6.9 (Gather Transposition)**.**

$$(\mathrm{G}_f^{N,n})^\top = \mathrm{S}_f^{N,n},$$

Property 6.10 (Scatter Transposition)**.**

$$(\mathrm{S}_f^{N,n})^\top = \mathrm{G}_f^{N,n}\,.$$

Multiplication of gather matrices with scatter matrices yields identity matrices.

Property 6.11 (Gather/Scatter Identity)**.**

$$\mathrm{G}_f^{N,n}\ \mathrm{S}_f^{N,n} = \mathrm{I}_n\,.$$

Property 6.12 (Scatter/Gather Identity).

$$\mathrm{S}_f^{N,n}\ \mathrm{G}_f^{N,n} = \mathrm{diag}\left(\delta_f(i)\right)$$

with the generating function

$$\delta_f : \begin{cases} \{0,...,N-1\} \to \mathbb{C} \\ i \mapsto \delta_f(i) \end{cases} \qquad \text{with} \quad \delta_f(i) = \begin{cases} 1 & \text{if } i \in f\left(\{0,\ldots,n-1\}\right) \\ 0 & \text{else} \end{cases}$$

A product of two gather matrices is a gather matrix.

Property 6.13 (Gather Multiplicativity).

$$\mathrm{G}_f^{N_1,n}\ \ \mathrm{G}_g^{N,N_1} = \mathrm{G}_{g\circ f}^{N,n}\ .$$

A product of two scatter matrices is a scatter matrix.

Property 6.14 (Scatter Multiplicativity).

$$\mathrm{S}_f^{N,N_1}\ \ \mathrm{S}_g^{N_1,n} = \mathrm{S}_{f\circ g}^{N,n}$$

A stack of two gather matrices is a gather matrix.

Property 6.15 (Gather Stacking).

$$\begin{bmatrix} \mathrm{G}_f^{N,n_1} \\ \mathrm{G}_g^{N,n_2} \end{bmatrix} = \mathrm{G}_{\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]}^{N,n_1+n_2}$$

A row of two scatter matrices is a scatter matrix.

Property 6.16 (Scatter Stacking).

$$\begin{bmatrix} \mathrm{S}_f^{N,n_1}\ \mathrm{S}_g^{N,n_2} \end{bmatrix} = \mathrm{S}_{\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]}^{N,n_1+n_2}$$

Property 6.17 (Gather/Scatter Multiplicativity).

$$\mathrm{G}_f^{N,n}\ \mathrm{S}_g^{N,n} = \mathrm{perm}(f\circ g) \quad \text{for} \quad \Lambda(f) = \Lambda(g).$$

Property 6.18 (Permutation as Gather Matrix).

$$\mathrm{G}_\pi^{N,N} = \mathrm{perm}(\pi) \quad \text{for} \quad \pi \in \mathrm{S}_N$$

Property 6.19 (Permutation as Scatter Matrix).

$$\mathrm{S}_\pi^{N,N} = \mathrm{perm}(\pi^{-1}) \quad \text{for} \quad \pi \in \mathrm{S}_N$$

Property 6.20 (Gather/Permutation Multiplicativity).

$$\mathrm{G}_f^{N,n}\ \mathrm{perm}(\pi) = \mathrm{G}_{\pi\circ f}^{N,n} \quad \text{for} \quad \pi \in \mathrm{S}_N\ .$$

Property 6.21 (Permutation/Gather Multiplicativity).

$$\mathrm{perm}(\pi)\ \mathrm{G}_f^{N,n} = \mathrm{G}_{f\circ\pi}^{N,n} \quad \text{for} \quad \pi \in \mathrm{S}_n\ .$$

Property 6.22 (Scatter/Permutation Multiplicativity).

$$\mathrm{S}_f^{N,n}\ \mathrm{perm}(\pi) = \mathrm{S}_{f\circ\pi^{-1}}^{N,n} \quad \text{for} \quad \pi \in \mathrm{S}_n\ .$$

Property 6.23 (Permutation/Scatter Multiplicativity).

$$\mathrm{perm}(\pi)\ \mathrm{S}_f^{N,n} = \mathrm{S}_{\pi^{-1}\circ f}^{N,n} \quad \text{for} \quad \pi \in \mathrm{S}_N\ .$$

Theorem 6.5 (Index Mapping Decomposition)**.** The matrix

$$\text{matrix}\left(\hat{f}\right)$$

generated by the matrix generation function induced by an index mapping function

$$f : \begin{cases} \{i_0, ..., i_{m-1}\} \to \{j_0, \dots, j_{m-1}\} \\ i \mapsto f(i) \end{cases}$$

with

$$f = r \circ w^{-1},$$

can be factored into a product of a scatter and gather matrix,

$$\mathrm{S}_w^{i_{m-1},m}\ \mathrm{G}_r^{j_{m-1},m},$$

with

$$r : \begin{cases} \{0, \dots, m-1\} \to \{j_0, \dots, j_{m-1}\} \\ i \mapsto r(i) \end{cases}$$

and

$$w : \begin{cases} \{0, \dots, m-1\} \to \{i_0, ..., i_{m-1}\} \\ i \mapsto w(i) \end{cases} \quad .$$

7 Expressing Constructs using Gather, Scatter and Iterative Sums

7.1 Iterative Constructs

The iterative constructs covered here are all translated into a iterative sum of k parametriced matrices A_j using the gather and scatter operators $\mathrm{G}_{r_j}^{N,n}$ and $\mathrm{S}_{w_j}^{M,m}$ parameterized by iteration dependent index mapping functions

$$r_j : \begin{cases} \{0, \dots, n-1\} \to \{0, \dots, N-1\} \\ i \mapsto r'(i,j) \end{cases}$$

and

$$w_j : \begin{cases} \{0, \dots, m-1\} \to \{0, \dots, M-1\} \\ i \mapsto w'(i,j) \end{cases}$$

leading to an equation

$$\text{construct}_{j=0}^{k-1} A_j = \sum_{j=0}^{k-1} \mathrm{S}_{i \mapsto w'(i,j)}^{M,m}\, A_j\, \mathrm{G}_{i \mapsto r'(i,j)}^{N,n}$$

Theorem 7.1 (Iterative Sum)**.**

$$\sum_{j=0}^{k-1} A_j = \sum_{j=0}^{k-1} \mathrm{S}_{i \mapsto i}^{mk,m}\, A_j\, \mathrm{G}_{i \mapsto i}^{nk,n} \quad \text{with} \quad A_j \in \mathbb{C}^{m \times n}$$

Theorem 7.2 (Iterative Direct Sum)**.**

$$\bigoplus_{j=0}^{k-1} A_j = \sum_{j=0}^{k-1} \mathrm{S}_{i \mapsto jm+i}^{mk,m}\, A_j\, \mathrm{G}_{i \mapsto jn+i}^{nk,n} \quad \text{with} \quad A_j \in \mathbb{C}^{m \times n}$$

Theorem 7.3 (Iterative Row Overlapped Direct Sum)**.**

$$\bigoplus_{j=0}^{k-1}{}_r A_j = \sum_{j=0}^{k-1} \mathrm{S}_{i \mapsto jm+i}^{mk,m}\, A_j\, \mathrm{G}_{i \mapsto j(n-r)+i}^{(n-r)k+r,n} \quad \text{with} \quad A_j \in \mathbb{C}^{m \times n}$$

Theorem 7.4 (Iterative Column Overlapped Direct Sum).

$$\bigoplus_{j=0}^{k-1}{}^{r} A_j = \sum_{j=0}^{k-1} \mathrm{S}_{i\mapsto j(m-r)+i}^{(m-r)k+r,m}\, A_j\, \mathrm{G}_{i\mapsto jn+i}^{nk,n} \quad \text{with} \quad A_j \in \mathbb{C}^{m\times n}$$

Theorem 7.5 (Parallel Tensor Product).

$$\mathrm{I}_k \otimes A = \sum_{j=0}^{k-1} \mathrm{S}_{i\mapsto jm+i}^{mk,m}\, A\, \mathrm{G}_{i\mapsto jn+i}^{nk,n} \quad \text{with} \quad A \in \mathbb{C}^{m\times n}$$

Theorem 7.6 (Row Overlapped Tensor Product).

$$\mathrm{I}_k \otimes_r A = \sum_{j=0}^{k-1} \mathrm{S}_{i\mapsto jm+i}^{mk,m}\, A_j\, \mathrm{G}_{i\mapsto j(n-r)+i}^{(n-r)k+r,n} \quad \text{with} \quad A \in \mathbb{C}^{m\times n}$$

Theorem 7.7 (Column Overlapped Tensor Product).

$$\mathrm{I}_k \otimes^r A = \sum_{j=0}^{k-1} \mathrm{S}_{i\mapsto j(m-r)+i}^{(m-r)k+r,m}\, A_j\, \mathrm{G}_{i\mapsto jn+i}^{nk,n} \quad \text{with} \quad A \in \mathbb{C}^{m\times n}$$

Theorem 7.8 (Vector Tensor Product).

$$A \otimes \mathrm{I}_k = \sum_{j=0}^{k-1} \mathrm{S}_{i\mapsto j+ik}^{mk,m}\, A\, \mathrm{G}_{i\mapsto j+ik}^{nk,n} \quad \text{with} \quad A \in \mathbb{C}^{m\times n}$$

7.2 Parametrized Constructs

7.2.1 Matrices of Matrices

Theorem 7.9 (Matrix of Matrices).

$$\begin{bmatrix} A_{0,0} & \cdots & A_{0,S-1} \\ \vdots & \ddots & \vdots \\ A_{R-1,0} & \cdots & A_{R-1,S-1} \end{bmatrix} = \sum_{j=0}^{R-1}\sum_{k=0}^{S-1} \mathrm{S}_{i\mapsto jm+i}^{mR,m}\, A_{j,k}\, \mathrm{G}_{i\mapsto kn+i}^{nS,n} \quad \text{with} \quad A_{j,k} \in \mathbb{C}^{m\times n}$$

Corollary 7.1 (Horizontal Stack of Matrices).

$$\Big[\ \Big|\ \Big]_{j=0}^{S-1} A_j = \sum_{j=0}^{S-1} \mathrm{S}_{\mathrm{id}}^{m,m}\, A_j\, \mathrm{G}_{i\mapsto jn+i}^{nS,n} \quad \text{with} \quad A_j \in \mathbb{C}^{m\times n}$$

Corollary 7.2 (Vertical Stack of Matrices).

$$\Big[\!-\!\Big]_{j=0}^{R-1} A_j = \sum_{j=0}^{R-1} \mathrm{S}_{i\mapsto jm+i}^{Rm,m}\, A_j\, \mathrm{G}_{\mathrm{id}}^{n,n} \quad \text{with} \quad A_j \in \mathbb{C}^{m\times n}$$

7.2.2 Diagonal Matrices

Theorem 7.10 (Diagonal Matrices). For any family of interval mapping functions

$$\{b_j\}_{j=0,\ldots,k-1} \quad \text{with} \quad b_j : \begin{cases} \{0,\ldots,n_j-1\} \to \{0,\ldots,n-1\} \\ i \mapsto b_j(i) \end{cases}$$

that factors the identity by

$$\mathrm{id}_n = \bigcup_{j=0}^{k-1} b_j b_j^{-1},$$

a diagonal matrix

$$\mathrm{diag}(f) \in \mathbb{C}^{n\times n} \quad \text{with} \quad f : \begin{cases} \{0,\dots,n-1\} \to \mathbb{C} \\ i \mapsto f(i) \end{cases}$$

can be expressed as sum with k iterations by

$$\mathrm{diag}(f) = \sum_{j=0}^{k-1} \mathrm{S}^{n,n_j}_{b(j)} \,\mathrm{diag}(f_j)\, \mathrm{G}^{n,n_j}_{b(j)} \quad \text{with} \quad f_j = f \circ b_j.$$

Corollary 7.3 (Diagonal Matrices)**.** A diagonal matrix

$$\mathrm{diag}(f) \in \mathbb{C}^{n\times n} \quad \text{with} \quad f : \begin{cases} \{0,\dots,n-1\} \to \mathbb{C} \\ i \mapsto f(i) \end{cases}$$

can be expressed as sum with k iterations $(k|n)$ by

$$\mathrm{diag}(f) = \sum_{j=0}^{k-1} \mathrm{S}^{n,\frac{n}{k}}_{i\mapsto j\frac{n}{k}+i} \,\mathrm{diag}(f_j)\, \mathrm{G}^{n,\frac{n}{k}}_{i\mapsto j\frac{n}{k}+i} \quad \text{with} \quad f_j : \begin{cases} \left\{0,\dots,\frac{n}{k}-1\right\} \to \mathbb{C} \\ i \mapsto f\left(j\frac{n}{k}+i\right) \end{cases}$$

Corollary 7.4 (Diagonal Matrices)**.** A diagonal matrix

$$\mathrm{diag}(f) \in \mathbb{C}^{n\times n} \quad \text{with} \quad f : \begin{cases} \{0,\dots,n-1\} \to \mathbb{C} \\ i \mapsto f(i) \end{cases}$$

can be expressed as sum with k iterations $(k|n)$ by

$$\mathrm{diag}(f) = \sum_{j=0}^{k-1} \mathrm{S}^{n,\frac{n}{k}}_{i\mapsto j+ik} \,\mathrm{diag}(f_j)\, \mathrm{G}^{n,\frac{n}{k}}_{i\mapsto j+ik} \quad \text{with} \quad f_j : \begin{cases} \left\{0,\dots,\frac{n}{k}-1\right\} \to \mathbb{C} \\ i \mapsto f(j+ik) \end{cases}$$

7.3 Permutations

7.3.1 Permutation Generating Functions

Theorem 7.11 (Domain Splitting of Permutation Generating Functions)**.** A permutation generating function

$$\pi : \begin{cases} \{0,\dots,n-1\} \to \{0,\dots,n-1\} \\ i \mapsto \pi(i) \end{cases} \qquad \text{with} \quad \pi \in \mathrm{S}_n$$

is split into the family of index mapping functions

$$\{\pi_j\}_{j=0,\dots,k-1} \quad , \quad \pi_j : \begin{cases} I_j \to J \\ i \mapsto \pi_j(i) \end{cases}$$

with

$$\pi = \bigcup_{j=0}^{k-1} \pi_j$$

by partitioning the domain of π into the domains of π_j,

$$\{0,\dots,n-1\} = \bigcup_{j=0}^{k-1} I_j \quad , \quad I_k \cap I_l = \varnothing \text{ for } k \neq l,$$

and defining

$$\pi_j := \pi|_{I_j}.$$

Theorem 7.12 (Range Splitting of Permutation Generating Functions)**.** A permutation generating function

$$\pi : \begin{cases} \{0, \dots, n-1\} \to \{0, \dots, n-1\} \\ i \mapsto \pi(i) \end{cases} \qquad \text{with} \quad \pi \in \mathrm{S}_n$$

is split into the family of index mapping functions

$$\{\pi_j\}_{j=0,\dots,k-1} \quad , \quad \pi_j : \begin{cases} I_j \to J \\ i \mapsto \pi_j(i) \end{cases}$$

with

$$\pi = \bigcup_{j=0}^{k-1} \pi_j$$

by partitioning the range of π into the domains of π_j,

$$\{0, \dots, n-1\} = \bigcup_{j=0}^{k-1} J_j \quad , \quad J_k \cap J_l = \varnothing \text{ for } k \neq l,$$

and defining

$$\pi_j := \left(\pi^{-1}|_{J_j}\right)^{-1}.$$

Theorem 7.13 (Decomposition into Interval Mapping Functions)**.** Any permutation generation function

$$\pi : \begin{cases} \{0, \dots, n-1\} \to \{0, \dots, n-1\} \\ i \mapsto \pi(i) \end{cases} \qquad \text{with} \quad \pi \in \mathrm{S}_n$$

can be decomposed into a fusion of concatenations of two families of interval mapping functions,

$$\pi = \bigcup_{j=0}^{k-1} r_j \circ w_j^{-1},$$

with

$$r_j : \begin{cases} \{0, \dots, m_j - 1\} \to \{0, \dots, n-1\} \\ i \mapsto r'(i, j) \end{cases}$$

and

$$w_j : \begin{cases} \{0, \dots, m_j - 1\} \to \{0, \dots, n-1\} \\ i \mapsto w'(i, j) \end{cases} \qquad .$$

Theorem 7.14 (Decomposition w.r.t. Interval Mapping Functions)**.** For any family of interval mapping functions

$$\left\{b_j\right\}_{j=0,\dots,k-1} \quad \text{with} \quad b_j : \begin{cases} \{0, \dots, n_j - 1\} \to \{0, \dots, n-1\} \\ i \mapsto b_j(i) \end{cases}$$

that factors the identity by

$$\mathrm{id}_n = \bigcup_{j=0}^{k-1} b_j b_j^{-1},$$

a permutation matrix

$$\pi : \begin{cases} \{0, \dots, n-1\} \to \{0, \dots, n-1\} \\ i \mapsto \pi(i) \end{cases} \qquad \text{with} \quad \pi \in \mathrm{S}_n$$

can be expressed as

$$\pi = \bigcup_{j=0}^{k-1} b_j \left(b_j^{-1} \circ \pi\right)$$

and

$$\pi = \bigcup_{j=0}^{k-1} (\pi \circ b_j)\, b_j^{-1}.$$

Definition 156 (Input Locality)**.** A split permutation generation function

$$\pi = \bigcup_{j=0}^{k-1} r_j \circ w_j^{-1} \quad \text{with} \quad \pi \in \mathrm{S}_n$$

and the corresponding interval mapping family

$$\{r_j\}_{j=0,\dots,m-1} \quad \text{with} \quad r_j : \begin{cases} \{0,\dots,n_j-1\} \to \{0,\dots,N-1\} \\ i \mapsto r_j(i) \end{cases}$$

has input locality

$$\Lambda\left(\{r_j\}_{j=0,\dots,m-1}\right).$$

Definition 157 (Output Locality)**.** A split permutation generation function

$$\pi = \bigcup_{j=0}^{k-1} r_j \circ w_j^{-1} \quad \text{with} \quad \pi \in \mathrm{S}_n$$

and the corresponding interval mapping family

$$\{w_j\}_{j=0,\dots,m-1} \quad \text{with} \quad w_j : \begin{cases} \{0,\dots,n_j-1\} \to \{0,\dots,N-1\} \\ i \mapsto r_j(i) \end{cases}$$

has output locality

$$\Lambda\left(\{w_j\}_{j=0,\dots,m-1}\right).$$

Definition 158 (Additive **k**-Separability w.r.t. Input Locality)**.** A permutation generation function

$$\pi \quad \text{with} \quad \pi \in \mathrm{S}_n : \begin{cases} \{0,\dots,n-1\} \to \{0,\dots,n-1\} \\ i \mapsto \pi(i) \end{cases}$$

is additive k-separable with respect to an input locality

$$\Phi = \left\{I_j\right\}_{j=0,\dots,m-1}$$

if π can be factorized into two additive k-separable families, i. e.,

$$\pi = \bigcup_{j=0}^{m-1} r_j \circ w_j^{-1}$$

and if $\{r_j\}_{j=0\dots,m-1}$ and $\{w_j\}_{j=0\dots,m-1}$ are additive k-separable and

$$\Lambda\Big(\left\{r_j\right\}_{j=0\dots,m-1}\Big) = \Phi.$$

Definition 159 (Additive **k**-Separability w.r.t. Output Locality)**.** A permutation generation function

$$\pi \quad \text{with} \quad \pi \in \mathrm{S}_n : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, n-1\} \\ i \mapsto \pi(i) \end{cases}$$

is additive k-separable with respect to an output locality

$$\Psi = \left\{ I_j \right\}_{j=0,\ldots,m-1}$$

if π can be factorized into two additive k-separable families, i. e.,

$$\pi = \bigcup_{j=0}^{m-1} r_j \circ w_j^{-1} \quad \text{with}$$

and

$$\Lambda\Big(\left\{ w_j \right\}_{j=0\ldots,m-1} \Big) = \Psi.$$

Definition 160 (Additive **k**-Separability)**.** A permutation generation function

$$\pi \quad \text{with} \quad \pi \in \mathrm{S}_n : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, n-1\} \\ i \mapsto \pi(i) \end{cases}$$

is additive k-separable if an input locality Φ exists that π is additive k-separable with respect to input locality Φ or if an output locality Ψ exists that π is additive k-separable with respect to output locality Ψ.

Definition 161 (Additive Separability w.r.t. Input Locality)**.** A permutation generation function

$$\pi \quad \text{with} \quad \pi \in \mathrm{S}_n : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, n-1\} \\ i \mapsto \pi(i) \end{cases}$$

is additive separable with respect to an input locality

$$\Phi = \left\{ I_j \right\}_{j=0,\ldots,m-1}$$

if it is additive 1-separable with respect to input locality Φ.

Definition 162 (Additive Separability w.r.t. Output Locality)**.** A permutation generation function

$$\pi \quad \text{with} \quad \pi \in \mathrm{S}_n : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, n-1\} \\ i \mapsto \pi(i) \end{cases}$$

is additive separable with respect to an output locality

$$\Psi = \left\{ I_j \right\}_{j=0,\ldots,m-1}$$

if it is additive 1-separable with respect to output locality Ψ.

Definition 163 (Additive Separability)**.** A permutation generation function

$$\pi \quad \text{with} \quad \pi \in \mathrm{S}_n : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, n-1\} \\ i \mapsto \pi(i) \end{cases}$$

is additive separable if an input locality Φ exists that π is additive separable with respect to input locality Φ or if an output locality Ψ exists that π is additive separable with respect to output locality Ψ.

7.3.2 Splitting Permutations into Iterative Sums

Theorem 7.15 (Permutation Splitting)**.** The permutation matrix

$$\mathrm{perm}\left(\pi\right)$$

generated by an permutation generating function

$$\pi : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, n-1\} \\ i \mapsto \pi(i) \end{cases}$$

with

$$\pi = \bigcup_{j=0}^{m-1} r_j \circ w_j^{-1} \qquad \pi \in \mathrm{S}_n$$

can be expressed by the iterative sum

$$\mathrm{perm}\left(\pi\right) = \sum_{j=0}^{m-1} \mathrm{S}_{i \mapsto w_j(i)}^{n,m_j} \; \mathrm{G}_{i \mapsto r_j(i)}^{n,m_j} .$$

Theorem 7.16 (Permutation Additive **k**-Splitting)**.** The permutation matrix

$$\mathrm{perm}\left(\pi\right)$$

generated by an permutation generating function

$$\pi : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, n-1\} \\ i \mapsto \pi(i) \end{cases}$$

with

$$\pi = \bigcup_{j=0}^{m-1} r_j \circ w_j^{-1} \qquad \pi \in \mathrm{S}_n$$

that is additive k-separable can be expressed by the iterative sum

$$\mathrm{perm}\left(\pi\right) = \sum_{j=0}^{m-1} \mathrm{S}_{i \mapsto b^w(j) + s^w_{\iota(j)}(i)}^{n,m_{\iota(j)}} \; \mathrm{G}_{i \mapsto b^r(j) + s^r_{\iota(j)}(i)}^{n,m_{\iota(j)}} .$$

Theorem 7.17 (Permutation Additive Splitting)**.** The permutation matrix

$$\mathrm{perm}\left(\pi\right)$$

generated by an permutation generating function

$$\pi : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, n-1\} \\ i \mapsto \pi(i) \end{cases}$$

with

$$\pi = \bigcup_{j=0}^{m-1} r_j \circ w_j^{-1} \qquad \pi \in \mathrm{S}_n$$

that is additive separable can be expressed by the iterative sum

$$\mathrm{perm}\left(\pi\right) = \sum_{j=0}^{m-1} \mathrm{S}_{i \mapsto b^w(j) + s^w(i)}^{n,m} \; \mathrm{G}_{i \mapsto b^r(j) + s^r(i)}^{n,m} .$$

Theorem 7.18 (Permutation Linear Splitting)**.** The permutation matrix

$$\mathrm{perm}\left(\pi\right)$$

generated by an permutation generating function

$$\pi : \begin{cases} \{0,\dots,n-1\} \rightarrow \{0,\dots,n-1\} \\ i \mapsto \pi(i) \end{cases}$$

with

$$\pi = \bigcup_{j=0}^{m-1} r_j \circ w_j^{-1} \qquad \pi \in \mathrm{S}_n$$

that is linear separable can be expressed by the iterative sum

$$\mathrm{perm}\left(\pi\right) = \sum_{j=0}^{m-1} \mathrm{S}^{n,m}_{i \mapsto u^w + v^w j + s^w(i)}\ \mathrm{G}^{n,m}_{i \mapsto u^r + v^r j + s^r(i)}\,.$$

7.3.3 Basic Permutations

Definition 164 (Identity Permutation Generating Function)**.** The identity permutation

$$\mathrm{I}_n = \mathrm{perm}\left(\imath_n\right)$$

is generated by

$$\imath_n : \begin{cases} \{0,\dots,n-1\} \rightarrow \{0,\dots,n-1\} \\ i \mapsto i \end{cases}\quad.$$

Theorem 7.19 (Linear Separability of Identity Permutations)**.** The identity permutation

$$\mathrm{I}_n = \mathrm{perm}\left(\imath_n\right)$$

is for all $k|n$ linear separable with respect to the input locality

$$\Phi_k = \left\{\left\{j\frac{n}{k} + i : i = 0,\dots,\frac{n}{k} - 1\right\} : j = 0,\dots,k-1\right\}$$

and output locality

$$\Psi_k = \left\{\left\{j\frac{n}{k} + i : i = 0,\dots,\frac{n}{k} - 1\right\} : j = 0,\dots,k-1\right\}.$$

Corollary 7.5 (Identity Permutation Splitting)**.** The identity permutation

$$\mathrm{I}_n = \mathrm{perm}\left(\imath_n\right)$$

can be expressed by the iterative sum

$$\mathrm{I}_n = \sum_{j=0}^{k-1} \mathrm{S}^{n,\frac{n}{k}}_{i \mapsto j\frac{n}{k}+i}\ \mathrm{G}^{n,\frac{n}{k}}_{i \mapsto j\frac{n}{k}+i} \quad , \quad k|n.$$

Lemma 7.1. The identity permutation generating function $\imath_n$ can be factored into

$$\imath_n = \bigcup_{j=0}^{k-1} r_j \circ w_j^{-1} \quad , \quad k|n, \tag{2}$$

with

$$r_j : \begin{cases} \left\{0,\dots,\frac{n}{k} - 1\right\} \rightarrow \{0,\dots,n-1\} \\ i \mapsto j\frac{n}{k} + i \end{cases}$$

and

$$w_j : \begin{cases} \left\{0,\dots,\frac{n}{k} - 1\right\} \rightarrow \{0,\dots,n-1\} \\ i \mapsto j\frac{n}{k} + i \end{cases}\quad.$$

Definition 165 (Opposite Diagonal Permutation Generating Function)**.** The opposite diagonal permutation

$$\mathrm{J}_n = \mathrm{perm}\left(\jmath_n\right)$$

is generated by

$$\jmath_n : \begin{cases} \{0,\dots,n-1\} \rightarrow \{0,\dots,n-1\} \\ i \mapsto n-1-i \end{cases} .$$

Theorem 7.20 (Linear Separability of Opposite Diagonal Permutations)**.** The opposite diagonal permutation

$$\mathrm{J}_n = \mathrm{perm}\left(\jmath_n\right)$$

is for all $k|n$ linear separable with respect to the input locality

$$\Phi_k = \left\{\left\{j\frac{n}{k} + i : i = 0,\dots,\frac{n}{k} - 1\right\} : j = 0,\dots,k-1\right\}$$

and output locality

$$\Psi_k = \left\{\left\{j\frac{n}{k} + i : i = 0,\dots,\frac{n}{k} - 1\right\} : j = 0,\dots,k-1\right\}.$$

Corollary 7.6 (Opposite Diagonal Permutation Splitting)**.** The opposite diagonal permutation

$$\mathrm{J}_n = \mathrm{perm}\left(\jmath_n\right)$$

can be expressed by the iterative sums

$$\mathrm{J}_n = \sum_{j=0}^{k-1} \mathrm{S}^{n,\frac{n}{k}}_{i\mapsto j\frac{n}{k}+i}\ \mathrm{G}^{n,\frac{n}{k}}_{i\mapsto (n-1)-j\frac{n}{k}-i} \quad , \quad k|n$$

and

$$\mathrm{J}_n = \sum_{j=0}^{k-1} \mathrm{S}^{n,\frac{n}{k}}_{i\mapsto (n-1)-j\frac{n}{k}-i}\ \mathrm{G}^{n,\frac{n}{k}}_{i\mapsto j\frac{n}{k}+i} \quad , \quad k|n.$$

Lemma 7.2. The opposite diagonal generating function $\jmath_n$ can be factored into

$$\jmath_n = \bigcup_{j=0}^{k-1} r_j \circ w_j^{-1} \quad , \quad k|n, \tag{3}$$

with

$$r_j : \begin{cases} \left\{0,\dots,\frac{n}{k}-1\right\} \rightarrow \{0,\dots,n-1\} \\ i \mapsto (n-1) - j\frac{n}{k} - i \end{cases}$$

and

$$w_j : \begin{cases} \left\{0,\dots,\frac{n}{k}-1\right\} \rightarrow \{0,\dots,n-1\} \\ i \mapsto j\frac{n}{k} + i \end{cases} .$$

Lemma 7.3. The opposite diagonal generating function $\jmath_n$ can be factored into

$$\jmath_n = \bigcup_{j=0}^{k-1} r_j \circ w_j^{-1} \quad , \quad k|n, \tag{4}$$

with

$$r_j : \begin{cases} \left\{0,\dots,\frac{n}{k}-1\right\} \rightarrow \{0,\dots,n-1\} \\ i \mapsto j\frac{n}{k} + i \end{cases} .$$

and

$$w_j : \begin{cases} \left\{0,\dots,\frac{n}{k}-1\right\} \rightarrow \{0,\dots,n-1\} \\ i \mapsto (n-1) - j\frac{n}{k} - i \end{cases}$$

Lemma 7.4. Factorizations (2)-(4) have input locality

$$\Lambda\Big(\big\{r_j\big\}_{j=0\dots,m-1}\Big) = \Big\{\Big\{j\frac{n}{k}+i : i=0,\dots,\frac{n}{k}-1\Big\} : j=0,\dots,k-1\Big\}$$

and output locality

$$\Lambda\Big(\big\{w_j\big\}_{j=0\dots,m-1}\Big) = \Big\{\Big\{j\frac{n}{k}+i : i=0,\dots,\frac{n}{k}-1\Big\} : j=0,\dots,k-1\Big\}.$$

7.3.4 Stride Permutations

Definition 166 (Stride Permutation Generating Function)**.** The stride permutation

$$\mathrm{L}_m^{mn} = \mathrm{perm}\left(\ell_m^{mn}\right)$$

is generated by

$$\ell_m^{mn} : \begin{cases} \{0,\dots,mn-1\} \to \{0,\dots,mn-1\} \\ i \mapsto \begin{cases} (im) \bmod (mn-1) & \text{if } i < mn-1 \\ mn-1 & \text{if } i = mn-1 \end{cases} \end{cases} .$$

Corollary 7.7 (Stride Permutation Generating Function)**.** The stride permutation

$$\mathrm{L}_m^{mn} = \mathrm{perm}\left(\ell_m^{mn}\right)$$

is generated by

$$\ell_m^{mn} : \begin{cases} \{0,\dots,mn-1\} \to \{0,\dots,mn-1\} \\ i \mapsto \left\lfloor \frac{i}{n} \right\rfloor + m(i \bmod n) \end{cases} .$$

Theorem 7.21 (Linear Separability of Stride Permutations)**.** The stride permutation

$$\mathrm{L}_m^{mn} = \mathrm{perm}\left(\ell_m^{mn}\right)$$

is linear separable with respect to the input localities

$$\begin{aligned} \Phi_0 &= \big\{\{j+im : i=0,\dots,n-1\} : j=0,\dots,m-1\big\} \quad \text{as well as} \\ \Phi_1 &= \big\{\{jm+i : i=0,\dots,m-1\} : j=0,\dots,n-1\big\} \end{aligned}$$

and output localities

$$\begin{aligned} \Psi_0 &= \big\{\{jn+i : i=0,\dots,n-1\} : j=0,\dots,m-1\big\} \quad \text{as well as} \\ \Psi_1 &= \big\{\{j+in : i=0,\dots,m-1\} : j=0,\dots,n-1\big\}. \end{aligned}$$

Corollary 7.8 (Stride Permutation Splitting)**.** The stride permutation

$$\mathrm{L}_m^{mn} = \mathrm{perm}\left(\ell_m^{mn}\right)$$

can be expressed by the iterative sums

$$\mathrm{L}_m^{mn} = \sum_{j=0}^{m-1} \mathrm{S}_{i\mapsto jn+i}^{mn,n}\ \mathrm{G}_{i\mapsto j+im}^{mn,n}$$

and

$$\mathrm{L}_m^{mn} = \sum_{j=0}^{n-1} \mathrm{S}_{i\mapsto j+in}^{mn,m}\ \mathrm{G}_{i\mapsto jm+i}^{mn,m}.$$

Lemma 7.5. The stride permutation generating function ℓ_m^{mn} can be factored into

$$\ell_m^{mn} = \bigcup_{j=0}^{m-1} r_j \circ w_j^{-1} \tag{5}$$

with

$$r_j : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, mn-1\} \\ i \mapsto j + im \end{cases}$$

and

$$w_j : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, mn-1\} \\ i \mapsto jn + i \end{cases} \quad .$$

Lemma 7.6. Factorization (5) has input locality

$$\Lambda\Big(\{r_j\}_{j=0\ldots,m-1}\Big) = \big\{\{j + im : i = 0, \ldots, n-1\} : j = 0, \ldots, m-1\big\}$$

and output locality

$$\Lambda\Big(\{w_j\}_{j=0\ldots,m-1}\Big) = \big\{\{jn + i : i = 0, \ldots, n-1\} : j = 0, \ldots, m-1\big\}.$$

Lemma 7.7. The stride permutation generating function ℓ_m^{mn} can be factored into

$$\ell_m^{mn} = \bigcup_{j=0}^{n-1} r_j \circ w_j^{-1} \tag{6}$$

with

$$r_j : \begin{cases} \{0, \ldots, m-1\} \to \{0, \ldots, mn-1\} \\ i \mapsto jm + i \end{cases}$$

and

$$w_j : \begin{cases} \{0, \ldots, m-1\} \to \{0, \ldots, mn-1\} \\ i \mapsto j + in \end{cases} \quad .$$

Lemma 7.8. Factorization (6) has input locality

$$\Lambda\Big(\{r_j\}_{j=0\ldots,m-1}\Big) = \big\{\{jm + i : i = 0, \ldots, m-1\} : j = 0, \ldots, n-1\big\}$$

and output locality

$$\Lambda\Big(\{w_j\}_{j=0\ldots,m-1}\Big) = \big\{\{j + in : i = 0, \ldots, m-1\} : j = 0, \ldots, n-1\big\}.$$

7.3.5 Affine Permutations

Definition 167 (Affine Permutation Generating Function)**.** The affine permutation

$$\mathrm{A}_{a,b}^n = \mathrm{perm}\left(\alpha_{a,b}^n\right) \quad , \quad a \nmid n$$

is generated by

$$\alpha_a^n : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, n-1\} \\ i \mapsto ai + b \bmod n \end{cases} \quad , \quad a \nmid n.$$

Corollary 7.9 (Affine Permutation Generating Function)**.** The affine permutation

$$\mathrm{A}_a^n = \mathrm{perm}\left(\alpha_a^n\right) \quad , \quad a \nmid n \quad , \quad a \mid n+1$$

is generated by

$$\mathrm{A}_a^n : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, n-1\} \\ i \mapsto \left\lfloor \frac{ai}{n+1} \right\rfloor + a(i \bmod \frac{n+1}{a}) \end{cases} \quad .$$

7.3.6 Multiplicative Structure Permutations

Definition 168 (Multiplicative Permutation Generating Function)**.** The multiplicative permutation

$$\mathrm{K}^n_{a,b} = \mathrm{perm}\left(\kappa^n_{a,b}\right) \quad , \quad n \text{ prime and } a \text{ primitive element in } \mathbb{Z}_n,$$

is generated by

$$\kappa^n_{a,b} : \begin{cases} \{0,\dots,n-2\} \to \{0,\dots,n-2\} \\ i \mapsto (ba^i \bmod n) - 1 \end{cases} .$$

8 Manipulation of Iterative Sums

8.1 Fusion of Compatible Sums

8.1.1 Fusing General Sums

In the following, the product

$$A\,B \tag{7}$$

of two iterative sums

$$A = \sum_{j=0}^{m-1} \mathrm{S}^{N_3,n_{3,j}}_{w_{A,j}}\, A_j\, \mathrm{G}^{N,n_{2,j}}_{r_{A,j}} \quad , \quad A_j \in \mathbb{C}^{n_{3,j}\times n_{2,j}}, A \in \mathbb{C}^{N_3\times N}$$

and

$$B = \sum_{k=0}^{m-1} \mathrm{S}^{N,n_{1,k}}_{w_{B,k}}\, B_k\, \mathrm{G}^{N_0,n_{0,k}}_{r_{B,k}} \quad , \quad B_k \in \mathbb{C}^{n_{1,k}\times n_{0,k}}, B \in \mathbb{C}^{N\times N_0}$$

is considered.

Theorem 8.1 (Fusion of Iterative Sums: $\boldsymbol{\Lambda}(\{\mathbf{f_j}\})$ Compatible)**.** If

$$\Lambda\Big(\left\{r_{A,j}\right\}_{j=0,\dots,m-1}\Big) = \Lambda\Big(\left\{w_{B,k}\right\}_{k=0,\dots,m-1}\Big)$$

the product (7) can be fused into a single iterative sum

$$A\,B = \sum_{j=0}^{m-1} \mathrm{S}^{N_3,n_{3,j}}_{w_{A,j}}\, A_j\, \mathrm{perm}\left(w^{-1}_{B,\pi(j)} \circ r_{A,j}\right) B_{\pi(j)}\, \mathrm{G}^{N_0,n_{0,\pi(j)}}_{r_{B,\pi(j)}}$$

with a permutation

$$\pi : \begin{cases} \{0,\dots,m-1\} \to \{0,\dots m-1\} \\ i \mapsto \pi(i) \end{cases} \quad , \quad \pi \in \mathrm{S}_m,$$

satisfying the condition

$$\Lambda\big(r_{A,j}\big) = \Lambda\big(w_{B,\pi(j)}\big) \quad \forall j = 0,\dots,m-1.$$

Theorem 8.2 (Fusion of Iterative Sums: $\boldsymbol{\Lambda}(\mathbf{f_j})$ Compatible)**.** If

$$\Lambda\big(r_{A,j}\big) = \Lambda\big(w_{B,j}\big) \quad \forall j = 0,\dots,m-1$$

the product (7) can be fused into a single iterative sum

$$A\,B = \sum_{j=0}^{m-1} \mathrm{S}^{N_3,n_{3,j}}_{w_{A,j}}\, A_j\, \mathrm{perm}\left(w^{-1}_{B,j} \circ r_{A,j}\right) B_j\, \mathrm{G}^{N_0,n_{0,j}}_{r_{B,j}} .$$

Theorem 8.3 (Fusion of Iterative Sums: $\mathbf{f}_{\pi(\mathbf{j})}$ Compatible)**.** If a permutation

$$\pi : \begin{cases} \{0,\dots,m-1\} \to \{0,\dots m-1\} \\ i \mapsto \pi(i) \end{cases} \quad , \quad \pi \in \mathrm{S}_m,$$

exists that

$$r_{A,j} = w_{B,\pi(j)} \quad \forall j = 0,\dots,m-1,$$

the product (7) can be fused into a single iterative sum

$$A\,B = \sum_{j=0}^{m-1} \mathrm{S}_{w_{A,j}}^{N_3,n_{3,j}}\, A_j B_{\pi(j)}\, \mathrm{G}_{r_{B,\pi(j)}}^{N_0,n_{0,\pi(j)}}\,.$$

Theorem 8.4 (Fusion of Iterative Sums: $\mathbf{f_j}$ Compatible)**.** If

$$r_{A,j} = w_{B,j} \quad \forall j = 0,\dots,m-1,$$

the product (7) can be fused into a single iterative sum

$$A\,B = \sum_{j=0}^{m-1} \mathrm{S}_{w_{A,j}}^{N_3,n_{3,j}}\, A_j B_j\, \mathrm{G}_{r_{B,j}}^{N_0,n_{0,j}}\,.$$

8.1.2 Fusing Sums With Additive Gather and Scatter

In the following, the product

$$A\,B \tag{8}$$

of two iterative sums

$$A = \sum_{j=0}^{m-1} \mathrm{S}_{w_{A,j}}^{N_3,n_{3,j}}\, A_j\, \mathrm{G}_{r_{A,j}}^{N,n} \quad , \quad A_j \in \mathbb{C}^{n_{3,j}\times n}, A \in \mathbb{C}^{N_3\times N} \tag{9}$$

and

$$B = \sum_{j=0}^{m-1} \mathrm{S}_{w_{B,j}}^{N,n}\, B_j\, \mathrm{G}_{r_{B,j}}^{N_0,n_{0,j}} \quad , \quad B_j \in \mathbb{C}^{n\times n_{0,j}}, B \in \mathbb{C}^{N\times N_0} \tag{10}$$

with additive separable functions

$$r_{A,j} : \begin{cases} \{0,\dots,n-1\} \to \{0,\dots,N-1\} \\ i \mapsto b(j) + s(i) \end{cases}$$

and

$$w_{B,j} : \begin{cases} \{0,\dots,n-1\} \to \{0,\dots,N-1\} \\ i \mapsto c(j) + t(i) \end{cases}$$

is considered.

Theorem 8.5 (Fusion of Iterative Sums: $\mathbf{\Lambda}(\{\mathbf{f_j}\})$ Compatible)**.** If

$$\Lambda\big(b(j)\big) = \Lambda\big(c(j)\big) \quad \text{and} \quad \Lambda\big(s(j)\big) = \Lambda\big(t(j)\big) \quad \forall j = 0,\dots,m-1$$

the product (8) can be fused into a single iterative sum

$$A\,B = \sum_{j=0}^{m-1} \mathrm{S}_{w_{A,j}}^{N_3,n_{3,j}}\, A_j\, \mathrm{perm}\left(t^{-1}\circ s\right) B_{\pi(j)}\, \mathrm{G}_{r_{B,\pi(j)}}^{N_0,n_{0,\pi(j)}}$$

with a permutation

$$\pi : \begin{cases} \{0, \dots, m-1\} \to \{0, \dots m-1\} \\ i \mapsto \pi(i) \end{cases} \quad , \quad \pi \in \mathrm{S}_m,$$

satisfying the condition

$$b(j) = c(\pi(j)) \quad \forall j = 0, \dots, m-1.$$

Theorem 8.6 (Fusion of Iterative Sums: $\mathbf{\Lambda(f_j)}$ Compatible)**.** If

$$b(j) = c(j) \quad \text{and} \quad \Lambda\big(s(j)\big) = \Lambda\big(t(j)\big) \quad \forall j = 0, \dots, m-1.$$

the product (8) can be fused into a single iterative sum

$$A\,B = \sum_{j=0}^{m-1} \mathrm{S}_{w_{A,j}}^{N_3, n_{3,j}}\, A_j \,\mathrm{perm}\left(t^{-1} \circ s\right) B_j \,\mathrm{G}_{r_{B,j}}^{N_0, n_{0,j}}\,.$$

Theorem 8.7 (Fusion of Iterative Sums: $\mathbf{f_{\pi(j)}}$ Compatible)**.** If

$$\Lambda\big(b(j)\big) = \Lambda\big(c(j)\big) \quad \text{and} \quad s(j) = t(j) \quad \forall j = 0, \dots, m-1.$$

then a permutation

$$\pi : \begin{cases} \{0, \dots, m-1\} \to \{0, \dots m-1\} \\ i \mapsto \pi(i) \end{cases} \quad , \quad \pi \in \mathrm{S}_m,$$

exists that the product (8) can be fused into a single iterative sum

$$A\,B = \sum_{j=0}^{m-1} \mathrm{S}_{w_{A,j}}^{N_3, n_{3,j}}\, A_j B_{\pi(j)} \,\mathrm{G}_{r_{B,\pi(j)}}^{N_0, n_{0,\pi(j)}}\,.$$

Theorem 8.8 (Fusion of Iterative Sums: $\mathbf{f_j}$ Compatible)**.** If

$$b(j) = c(j) \quad \text{and} \quad s(j) = t(j) \quad \forall j = 0, \dots, m-1.$$

the product (8) can be fused into a single iterative sum

$$A\,B = \sum_{j=0}^{m-1} \mathrm{S}_{w_{A,j}}^{N_3, n_{3,j}}\, A_j B_j \,\mathrm{G}_{r_{B,j}}^{N_0, n_{0,j}}\,.$$

8.1.3 Fusing Sums and Diagonals

Corollary 8.1 (Commuting Gather with Vertical Stack)**.** For

$$A_j \in \mathbb{C}^{m \times n} \quad , \quad 0 \le j < k$$

and

$$r : \begin{cases} \{0, \dots, l-1\} \to \{0, \dots, k-1\} \\ i \mapsto r(i) \end{cases}$$

it holds that

$$\mathrm{G}_{r \otimes \imath_m}^{km, lm} \left(\underset{j\,=\,0}{\overset{k\,-\,1}{\left[\text{—}\right]}} A_j \right) = \underset{j\,=\,0}{\overset{l\,-\,1}{\left[\text{—}\right]}} A_{r(j)}.$$

Corollary 8.2 (Commuting Gather with Overlapped Direct Sum)**.** For

$$A_j \in \mathbb{C}^{m\times n} \quad , \quad 0 \le j < k$$

and

$$r : \begin{cases} \{0,\dots,l-1\} \to \{0,\dots,k-1\} \\ i \mapsto r(i) \end{cases}$$

it holds that

$$\mathrm{G}^{km,lm}_{r\otimes\imath_m} \left(\bigoplus_{j=0}^{k-1}{}_t A_j \right) = \left(\bigoplus_{j=0}^{l-1} A_{r(j)} \right) \mathrm{G}^{k(n-t)+t,ln}_{r\otimes_t \imath_n} .$$

Corollary 8.3 (Commuting Gather with Direct Sum)**.** For

$$A_j \in \mathbb{C}^{m\times n} \quad , \quad 0 \le j < k$$

and

$$r : \begin{cases} \{0,\dots,l-1\} \to \{0,\dots,k-1\} \\ i \mapsto r(i) \end{cases}$$

it holds that

$$\mathrm{G}^{km,lm}_{r\otimes\imath_m} \left(\bigoplus_{j=0}^{k-1} A_j \right) = \left(\bigoplus_{j=0}^{l-1} A_{r(j)} \right) \mathrm{G}^{kn,ln}_{r\otimes \imath_n} .$$

Corollary 8.4 (Commuting Gather with Diagonals)**.** For

$$f : \begin{cases} \{0,\dots,N-1\} \to \mathbb{C} \\ i \mapsto f(i) \end{cases}$$

and

$$r : \begin{cases} \{0,\dots,n-1\} \to \{0,\dots,N-1\} \\ i \mapsto r(i) \end{cases}$$

it holds that

$$\mathrm{G}^{N,n}_r \operatorname{diag}(f) = \operatorname{diag}(f\circ r)\, \mathrm{G}^{N,n}_r .$$

Corollary 8.5 (Fusing Sums with Diagonals)**.** Any product

$$A = \left(\sum_{j=0}^{m-1} \mathrm{S}^{N,n_j}_{w_j} A_j\, \mathrm{G}^{N,n_j}_{r_j} \right) \operatorname{diag}(f)$$

with

$$f : \begin{cases} \{0,\dots,N-1\} \to \mathbb{C} \\ i \mapsto f(i) \end{cases}$$

can be written as

$$A = \sum_{j=0}^{m-1} \mathrm{S}^{N,n_j}_{w_j} A_j \operatorname{diag}(f_j)\, \mathrm{G}^{N,n_j}_{r_j}$$

with

$$f_j : \begin{cases} \{0,\dots,n_j-1\} \to \mathbb{C} \\ i \mapsto (f\circ r_j)(i) \end{cases} .$$

Corollary 8.6 (Commuting Scatter with Horizontal Stack)**.** For

$$A_j \in \mathbb{C}^{m\times n} \quad , \quad 0 \le j < k$$

and

$$w : \begin{cases} \{0,\dots,l-1\} \to \{0,\dots,k-1\} \\ i \mapsto w(i) \end{cases}$$

it holds that

$$\left(\mathop{\left[\,\middle|\,\right]}_{j=0}^{k-1} A_j \right) \mathrm{S}^{kn,ln}_{w\otimes \imath_n} = \mathop{\left[\,\middle|\,\right]}_{j=0}^{l-1} A_{w(j)}.$$

Corollary 8.7 (Commuting Scatter with Overlapped Direct Sum)**.** For

$$A_j \in \mathbb{C}^{m\times n} \quad , \quad 0 \le j < k$$

and

$$w : \begin{cases} \{0,\dots,l-1\} \to \{0,\dots,k-1\} \\ i \mapsto w(i) \end{cases}$$

it holds that

$$\left(\bigoplus_{j=0}^{k-1}{}^{t} A_j \right) \mathrm{S}^{kn,ln}_{w\otimes \imath_m} = \mathrm{S}^{k(n-t)+t,ln}_{w\otimes_t \imath_n} \left(\bigoplus_{j=0}^{l-1} A_{w(j)} \right).$$

Corollary 8.8 (Commuting Scatter with Direct Sum)**.** For

$$A_j \in \mathbb{C}^{m\times n} \quad , \quad 0 \le j < k$$

and

$$w : \begin{cases} \{0,\dots,l-1\} \to \{0,\dots,k-1\} \\ i \mapsto r(i) \end{cases}$$

it holds that

$$\left(\bigoplus_{j=0}^{k-1} A_j \right) \mathrm{S}^{kn,ln}_{w\otimes \imath_n} = \mathrm{S}^{km,lm}_{w\otimes \imath_m} \left(\bigoplus_{j=0}^{l-1} A_{w(j)} \right).$$

Corollary 8.9 (Commuting Scatter with Diagonals)**.** For

$$f : \begin{cases} \{0,\dots,N-1\} \to \mathbb{C} \\ i \mapsto f(i) \end{cases}$$

and

$$w : \begin{cases} \{0,\dots,n-1\} \to \{0,\dots,N-1\} \\ i \mapsto r(i) \end{cases}$$

it holds that

$$\mathrm{diag}(f)\ \mathrm{S}^{N,n}_{w} = \mathrm{S}^{N,n}_{w}\ \mathrm{diag}(f \circ w).$$

Corollary 8.10 (Fusing Diagonals with Sums)**.** Any product

$$A = \mathrm{diag}(f) \left(\sum_{j=0}^{m-1} \mathrm{S}^{N,n_j}_{w_j}\, A_j\, \mathrm{G}^{N,n_j}_{r_j} \right)$$

with

$$f : \begin{cases} \{0,\dots,N-1\} \to \mathbb{C} \\ i \mapsto f(i) \end{cases}$$

can be written as

$$A = \sum_{j=0}^{m-1} \mathrm{S}_{w_j}^{N,n_j} \operatorname{diag}(f_j) A_j \, \mathrm{G}_{r_j}^{N,n_j}$$

with

$$f_j : \begin{cases} \{0,\ldots,n_j-1\} \to \mathbb{C} \\ i \mapsto (f \circ w_j)(i) \end{cases} .$$

Example 8.1 (Fusing Diagonals with Tensor Products)**.** This example shows how to obtain a single iterative sum for the construct

$$(A_n \otimes \mathrm{I}_m) \operatorname{diag}(f) \quad \text{with} \quad A_n \in \mathbb{C}^{n \times n}$$

with

$$f : \begin{cases} \{0,\ldots,mn-1\} \to \mathbb{C} \\ i \mapsto f(i) \end{cases} .$$

Theorem 7.8 provides

$$A_n \otimes \mathrm{I}_m = \sum_{j=0}^{m-1} \mathrm{S}_{i \mapsto j+im}^{mn,n} \, A_n \, \mathrm{G}_{i \mapsto j+im}^{mn,n} \tag{11}$$

leading to

$$(A_n \otimes \mathrm{I}_m) \operatorname{diag}(f) = \left(\sum_{j=0}^{m-1} \mathrm{S}_{i \mapsto j+im}^{mn,n} \, A_n \, \mathrm{G}_{i \mapsto j+im}^{mn,n} \right) \operatorname{diag}(f)$$

Application of Theorem 8.5 introduces the diagonal generating function f_j given by

$$f_j : \begin{cases} \{0,\ldots,n-1\} \to \mathbb{C} \\ i \mapsto f(j+im) \end{cases}$$

and leads to

$$(A_n \otimes \mathrm{I}_m) \operatorname{diag}(f) = \sum_{j=0}^{m-1} \mathrm{S}_{i \mapsto j+im}^{mn,n} \, A_n \operatorname{diag}_{i=0}^{n-1} \big(f(j+im) \big) \, \mathrm{G}_{i \mapsto j+im}^{mn,n} .$$

8.1.4 Fusing Sums and Permutations

Theorem 8.9 (Fusing Sums with Permutations)**.** Any product

$$A = \left(\sum_{j=0}^{m-1} \mathrm{S}_{w_j}^{N,n_j} \, A_j \, \mathrm{G}_{r_j}^{N,n_j} \right) \operatorname{perm}(\pi)$$

with

$$\pi : \begin{cases} \{0,\ldots,N-1\} \to \{0,\ldots,N-1\} \\ i \mapsto \pi(i) \end{cases} \qquad \text{with} \quad \pi \in \mathrm{S}_N$$

can be written as

$$A = \sum_{j=0}^{m-1} \mathrm{S}_{w_j}^{N,n_j} \, A_j \, \mathrm{G}_{\pi \circ r_j}^{N,n_j} .$$

Theorem 8.10 (Fusing Permutations with Sums)**.** Any product

$$A = \operatorname{perm}(\pi) \left(\sum_{j=0}^{m-1} \mathrm{S}_{w_j}^{N,n_j} \, A_j \, \mathrm{G}_{r_j}^{N,n_j} \right)$$

with

$$\pi : \begin{cases} \{0, \ldots, N-1\} \rightarrow \{0, \ldots, N-1\} \\ i \mapsto \pi(i) \end{cases} \qquad \text{with} \quad \pi \in \mathrm{S}_N$$

can be written as

$$A = \sum_{j=0}^{m-1} \mathrm{S}^{N,n_j}_{\pi^{-1} \circ w_j} \, A_j \, \mathrm{G}^{N,n_j}_{r_j} \, .$$

Example 8.2 (Compatible Stride Permutation)**.** This example shows how to obtain a single iterative sum for the construct

$$(\mathrm{I}_m \otimes A_n) \, \mathrm{L}^{mn}_m \quad \text{with} \quad A_n \in \mathbb{C}^{n \times n}. \tag{12}$$

In construct (12) the tensor product exactly matches the stride permutation. Theorem 7.5 provides

$$\mathrm{I}_m \otimes A = \sum_{j=0}^{m-1} \mathrm{S}^{mn,n}_{i \mapsto jn+i} \, A \, \mathrm{G}^{mn,n}_{i \mapsto jn+i}$$

Corollary 9.1 provides

$$\mathrm{L}^{mn}_m = \mathrm{perm}\left(\ell^{mn}_m\right)$$

with

$$\ell^{mn}_m : \begin{cases} \{0, \ldots, mn-1\} \rightarrow \{0, \ldots, mn-1\} \\ i \mapsto \left\lfloor \frac{i}{n} \right\rfloor + m(i \bmod n) \end{cases} .$$

(12) thus is expressed by

$$(\mathrm{I}_m \otimes A_n) \, \mathrm{L}^{mn}_m = \left(\sum_{j=0}^{m-1} \mathrm{S}^{mn,n}_{i \mapsto jn+i} \, A_n \, \mathrm{G}^{mn,n}_{i \mapsto jn+i} \right) \mathrm{perm}\left(\ell^{mn}_m\right). \tag{13}$$

Applying Theorem 8.9 with

$$r_j : \begin{cases} \{0, \ldots, n-1\} \rightarrow \{0, \ldots, mn-1\} \\ i \mapsto jn+i \end{cases}$$

and

$$\pi = \ell^{mn}_m$$

to (13) leads to

$$\pi \circ r_j : \begin{cases} \{0, \ldots, n-1\} \rightarrow \{0, \ldots, mn-1\} \\ i \mapsto \left\lfloor \frac{jn+i}{n} \right\rfloor + m\big((jn+i) \bmod n\big) \end{cases} .$$

Lemma 6.9 provides

$$\left\lfloor \frac{jn+i}{n} \right\rfloor = j \quad \text{for} \quad 0 \leq i < n$$

and Lemma 6.12 provides

$$(jn+i) \bmod n = i \quad \text{for} \quad 0 \leq i < n.$$

Thus, simplification leads to

$$\pi \circ r_j : \begin{cases} \{0, \ldots, n-1\} \rightarrow \{0, \ldots, mn-1\} \\ i \mapsto j + mi \end{cases}$$

and finally

$$(\mathrm{I}_m \otimes A_n) \, \mathrm{L}^{mn}_m = \sum_{j=0}^{m-1} \mathrm{S}^{mn,n}_{i \mapsto jn+i} \, A_n \, \mathrm{G}^{mn,n}_{i \mapsto j+im} \, .$$

The family

$$\{\pi \circ r_j\} = \left\{ \begin{cases} \{0,\dots,n-1\} \to \{0,\dots,mn-1\} \\ i \mapsto j + im \end{cases} \right\}_{j=0,\dots,m-1}$$

is additive separable with

$$b(j) = j \quad \text{and} \quad s(i) = mi$$

and the family

$$\{w_j\} = \left\{ \begin{cases} \{0,\dots,n-1\} \to \{0,\dots,mn-1\} \\ i \mapsto jn + i \end{cases} \right\}_{j=0,\dots,m-1}$$

is additive separable with

$$b(j) = jn \quad \text{and} \quad s(i) = i.$$

Example 8.3 (Stride Permutation–Too Many Loop Iterations)**.** This example shows how to obtain a single iterative sum for the construct

$$(\mathrm{I}_{km} \otimes A_n)\, \mathrm{L}_k^{kmn} \quad \text{with} \quad A_n \in \mathbb{C}^{n\times n}. \tag{14}$$

In construct (14) the tensor product has too many iterations to match the stride permutation. Theorem 7.5 provides

$$\mathrm{I}_{km} \otimes A_n = \sum_{j=0}^{km-1} \mathrm{S}_{i\mapsto jn+i}^{kmn,n}\, A\, \mathrm{G}_{i\mapsto jn+i}^{kmn,n}$$

Corollary 9.1 provides

$$\mathrm{L}_k^{kmn} = \mathrm{perm}\left(\ell_k^{kmn}\right)$$

with

$$\ell_k^{kmn} : \begin{cases} \{0,\dots,kmn-1\} \to \{0,\dots,kmn-1\} \\ i \mapsto \left\lfloor \frac{i}{mn} \right\rfloor + k(i \bmod mn) \end{cases}.$$

(14) thus is expressed by

$$(\mathrm{I}_{km} \otimes A_n)\, \mathrm{L}_k^{kmn} = \left(\sum_{j=0}^{km-1} \mathrm{S}_{i\mapsto jn+i}^{kmn,n}\, A\, \mathrm{G}_{i\mapsto jn+i}^{kmn,n} \right) \mathrm{perm}\left(\ell_k^{kmn}\right). \tag{15}$$

Applying Theorem 8.9 with

$$r_j : \begin{cases} \{0,\dots,n-1\} \to \{0,\dots,kmn-1\} \\ i \mapsto jn + i \end{cases}$$

and

$$\pi = \ell_k^{kmn}$$

to (15) leads to

$$\pi \circ r_j : \begin{cases} \{0,\dots,n-1\} \to \{0,\dots,kmn-1\} \\ i \mapsto \left\lfloor \frac{jn+i}{mn} \right\rfloor + k\big((jn+i) \bmod mn\big) \end{cases}.$$

Using

$$\left\lfloor \frac{jn+i}{mn} \right\rfloor = \left\lfloor \frac{j}{m} + \frac{i}{mn} \right\rfloor$$

and Lemma 6.10 leads to

$$\left\lfloor \frac{jn+i}{mn} \right\rfloor = \left\lfloor \frac{j}{m} \right\rfloor \quad \text{for} \quad 0 \le i < n.$$

Lemma 6.11 provides

$$(jn+i) \bmod mn = (jn \bmod mn) + (i \bmod mn)$$

and Lemma 6.13 provides

$$jn \bmod mn = n(j \bmod m).$$

Thus, simplification leads to

$$\pi \circ r_j : \begin{cases} \{0,\dots,n-1\} \to \{0,\dots,kmn-1\} \\ i \mapsto i \mapsto \left\lfloor \frac{j}{m} \right\rfloor + kn(j \bmod m) + k(i \bmod mn) \end{cases}$$

and finally

$$(\mathrm{I}_{km} \otimes A_n)\, \mathrm{L}_k^{kmn} = \sum_{j=0}^{km-1} \mathrm{S}_{i \mapsto jn+i}^{kmn,n}\, A_n\, \mathrm{G}_{i \mapsto \left\lfloor \frac{j}{m} \right\rfloor + kn(j \bmod m) + k(i \bmod mn)}^{kmn,n}.$$

The family

$$\{\pi \circ r_j\} = \left\{ \begin{cases} \{0,\dots,n-1\} \to \{0,\dots,kmn-1\} \\ i \mapsto i \mapsto \left\lfloor \frac{j}{m} \right\rfloor + kn(j \bmod m) + k(i \bmod mn) \end{cases} \right\}_{j=0,\dots,km-1}$$

is additive separable with

$$b(j) = \left\lfloor \frac{j}{m} \right\rfloor + kn(j \bmod m) \quad \text{and} \quad s(i) = k(i \bmod mn)$$

and the family

$$\{w_j\} = \left\{ \begin{cases} \{0,\dots,n-1\} \to \{0,\dots,kmn-1\} \\ i \mapsto jn+i \end{cases} \right\}_{j=0,\dots,km-1}$$

is additive separable with

$$b(j) = jn \quad \text{and} \quad s(i) = i.$$

Example 8.4 (Stride Permutation–Not Enough Loop Iterations)**.** This example shows how to obtain a single iterative sum for the construct

$$(\mathrm{I}_m \otimes A_{kn})\, \mathrm{L}_{km}^{kmn} \quad \text{with} \quad A_{kn} \in \mathbb{C}^{kn \times kn}. \tag{16}$$

In construct (16) the tensor product has not enough iterations to match the stride permutation. Theorem 7.5 provides

$$\mathrm{I}_m \otimes A_{kn} = \sum_{j=0}^{m-1} \mathrm{S}_{i \mapsto jkn+i}^{kmn,kn}\, A_{kn}\, \mathrm{G}_{i \mapsto jkn+i}^{kmn,kn}$$

Corollary 9.1 provides

$$\mathrm{L}_{km}^{kmn} = \mathrm{perm}\left(\ell_{km}^{kmn}\right)$$

with

$$\ell_{km}^{kmn} : \begin{cases} \{0,\dots,kmn-1\} \to \{0,\dots,kmn-1\} \\ i \mapsto \left\lfloor \frac{i}{n} \right\rfloor + km(i \bmod n) \end{cases}.$$

(16) thus is expressed by

$$(\mathrm{I}_m \otimes A_{kn})\, \mathrm{L}_{km}^{kmn} = \left(\sum_{j=0}^{m-1} \mathrm{S}_{i \mapsto jkn+i}^{kmn,kn}\, A_{kn}\, \mathrm{G}_{i \mapsto jkn+i}^{kmn,kn} \right) \mathrm{perm}\left(\ell_{km}^{kmn}\right). \tag{17}$$

Applying Theorem 8.9 with

$$r_j : \begin{cases} \{0,\dots,kn-1\} \to \{0,\dots,kmn-1\} \\ i \mapsto jkn+i \end{cases}$$

and

$$\pi = \ell_{km}^{kmn}$$

to (17) leads to

$$\pi \circ r_j : \begin{cases} \{0, \ldots, kn-1\} \to \{0, \ldots, kmn-1\} \\ i \mapsto \left\lfloor \frac{jkn+i}{n} \right\rfloor + km\big((jkn+i) \bmod n\big) \end{cases} .$$

Using

$$\left\lfloor \frac{jkn+i}{n} \right\rfloor = \left\lfloor jk + \frac{i}{n} \right\rfloor$$

and Lemma 6.14 leads to

$$\left\lfloor \frac{jkn+i}{n} \right\rfloor = jk + \left\lfloor \frac{i}{n} \right\rfloor .$$

Lemma 6.11 provides

$$(jkn+i) \bmod n = (jkn \bmod n) + (i \bmod n)$$

and Lemma 6.15 provides

$$jkn \bmod n = 0.$$

Thus, simplification leads to

$$\pi \circ r_j : \begin{cases} \{0, \ldots, kn-1\} \to \{0, \ldots, kmn-1\} \\ i \mapsto jk + \left\lfloor \frac{i}{n} \right\rfloor + km(i \bmod n) \end{cases} .$$

and finally

$$(\mathrm{I}_m \otimes A_{kn}) \, \mathrm{L}_{km}^{kmn} = \sum_{j=0}^{m-1} \mathrm{S}_{i \mapsto jkn+i}^{kmn,kn} \, A_{kn} \, \mathrm{G}_{i \mapsto jk + \lfloor \frac{i}{n} \rfloor + km(i \bmod n)}^{kmn,kn} .$$

The family

$$\{\pi \circ r_j\} = \left\{ \begin{cases} \{0, \ldots, kn-1\} \to \{0, \ldots, kmn-1\} \\ i \mapsto jk + \left\lfloor \frac{i}{n} \right\rfloor + km(i \bmod n) \end{cases} \right\}_{j=0,\ldots,m-1}$$

is additive separable with

$$b(j) = jk \quad \text{and} \quad s(i) = \left\lfloor \frac{i}{n} \right\rfloor + km(i \bmod n)$$

and the family

$$\{w_j\} = \left\{ \begin{cases} \{0, \ldots, kn-1\} \to \{0, \ldots, kmn-1\} \\ i \mapsto jkn+i \end{cases} \right\}_{j=0,\ldots,m-1}$$

is additive separable with

$$b(j) = jkn \quad \text{and} \quad s(i) = i.$$

8.2 Exchanging Order of Iterative Sums

Theorem 8.11 (Horizontally Stacked Matrices)**.** For a horizontally stacked matrix

$$A = \left[\Big| \right]_{k=0}^{N-1} \sum_{j=0}^{M-1} \mathrm{S}_{w_j}^{Mm,m} \, A_{j,k} \, \mathrm{G}_{r_{j,k}}^{MNn,n} \quad \text{with} \quad A_{j,k} \in \mathbb{C}^{m \times n},$$

the horizontally stacking can be pulled in leading to

$$A = \sum_{j=0}^{M-1} \mathrm{S}_{w_j}^{Mm,m} \left[\Big| \right]_{k=0}^{N-1} A_{j,k} \, \mathrm{G}_{[-]_{k=0}^{N-1} r_{j,k}}^{MNn,Nn} .$$

Theorem 8.12 (Vertically Stacked Matrices)**.** For a vertically stacked matrix

$$A = \mathop{[-]}_{k=0}^{N-1} \sum_{j=0}^{M-1} \mathrm{S}_{w_{j,k}}^{MNm,m} \, A_{j,k} \, \mathrm{G}_{r_j}^{Mn,n} \quad \text{with} \quad A_{j,k} \in \mathbb{C}^{m \times n},$$

the vertically stacking can be pulled in leading to

$$A = \sum_{j=0}^{M-1} \mathrm{S}_{[-]_{k=0}^{N-1} w_{j,k}}^{MNm,m} \mathop{[-]}_{k=0}^{N-1} A_{j,k} \, \mathrm{G}_{r_j}^{Mn,Mn} \, .$$

8.3 Loop Unrolling

Theorem 8.13 (Loop Unrolling)**.** A iterative sum

$$A = \sum_{j=0}^{p-1} \mathrm{S}_{w_j}^{M,m_j} \, A_j \, \mathrm{G}_{r_j}^{N,n_j} \quad \text{with} \quad A_j \in \mathbb{C}^{m_j \times n_j}$$

is fully unrolled by

$$A = \mathrm{S}_{[-]_{j=0}^{p-1} w_j}^{M,\sum_{j=0}^{p-1} m_j} \left(\bigoplus_{j=0}^{p-1} A_j \right) \mathrm{G}_{[-]_{j=0}^{p-1} r_j}^{N,\sum_{j=0}^{p-1} n_j} \, .$$

Corollary 8.11 (Loop Unrolling)**.** If

$$\sum_{j=0}^{p-1} m_j = M \quad \text{and} \quad \sum_{j=0}^{p-1} n_j = N,$$

the iterative sum

$$A = \sum_{j=0}^{p-1} \mathrm{S}_{w_j}^{M,m_j} \, A_j \, \mathrm{G}_{r_j}^{N,n_j} \quad \text{with} \quad A_j \in \mathbb{C}^{m_j \times n_j}$$

is fully unrolled by

$$A = \mathrm{perm} \left([-]_{j=0}^{p-1} w_j \right)^{-1} \left(\bigoplus_{j=0}^{p-1} A_j \right) \mathrm{perm} \left([-]_{j=0}^{p-1} r_j \right) .$$

Theorem 8.14 (Splitting of Iterative Sums)**.** A given iterative sum

$$A = \sum_{j=0}^{p-1} \mathrm{S}_{w_j}^{M,m_j} \, A_j \, \mathrm{G}_{r_j}^{N,n_j} \quad \text{with} \quad A_j \in \mathbb{C}^{m_j \times n_j}$$

is split with respect to a partition of the p iterations

$$\{0, \ldots, p-1\} = \bigcup_{k=0}^{q-1} J_k \quad \text{with} \quad J_i \cap J_j = \varnothing \text{ for } i \neq j$$

by

$$A = \sum_{k=0}^{q-1} \sum_{j \in J_k} \mathrm{S}_{w_j}^{M,m_j} \, A_j \, \mathrm{G}_{r_j}^{N,n_j} \, .$$

Corollary 8.12 (Partial Loop Unrolling)**.** A given iterative sum

$$A = \sum_{j=0}^{p-1} \mathrm{S}_{w_j}^{M,m_j}\, A_j\, \mathrm{G}_{r_j}^{N,n_j} \quad \text{with} \quad A_j \in \mathbb{C}^{m_j \times n_j}$$

is partially unrolled with respect to a partition of the p iterations

$$\{0, \ldots, p-1\} = \bigcup_{k=0}^{q-1} J_k \quad \text{with} \quad J_i \cap J_j = \varnothing \text{ for } i \neq j$$

by

$$A = \sum_{k=0}^{q-1} \mathrm{S}_{[-]_{j \in J_k} w_j}^{M, \sum_{j \in J_k} m_j} \bigoplus_{j \in J_k} A_j\, \mathrm{G}_{[-]_{j \in J_k} r_j}^{N, \sum_{j \in J_k} n_j}.$$

Corollary 8.13 (Partial Loop Unrolling for Fixed Matrix Size)**.** A given iterative sum

$$A = \sum_{j=0}^{p-1} \mathrm{S}_{w_j}^{M,m}\, A_j\, \mathrm{G}_{r_j}^{N,n} \quad \text{with} \quad A_j \in \mathbb{C}^{m \times n}$$

is partially unrolled with respect to a partition of the p iterations

$$\{0, \ldots, p-1\} = \bigcup_{k=0}^{q-1} J_k \quad \text{with} \quad J_i \cap J_j = \varnothing \text{ for } i \neq j$$

by

$$A = \sum_{k=0}^{q-1} \mathrm{S}_{[-]_{j \in J_k} w_j}^{M, |J_k| m} \bigoplus_{j \in J_k} A_j\, \mathrm{G}_{[-]_{j \in J_k} r_j}^{N, |J_k| n}.$$

Corollary 8.14 (Partial k-fold Loop Unrolling)**.** If

$$p = qr,$$

a given iterative sum

$$A = \sum_{j=0}^{p-1} \mathrm{S}_{w_j}^{M,m}\, A_j\, \mathrm{G}_{r_j}^{N,n} \quad \text{with} \quad A_j \in \mathbb{C}^{m \times n}$$

is r-fold partially unrolled by

$$A = \sum_{k=0}^{q-1} \mathrm{S}_{[-]_{l=0}^{r-1} w_{kr+l}}^{M,rm} \bigoplus_{l=0}^{r-1} A_{kr+l}\, \mathrm{G}_{[-]_{l=0}^{r-1} r_{kr+l}}^{N,rn}$$

and q-fold partially unrolled by

$$A = \sum_{k=0}^{r-1} \mathrm{S}_{[-]_{l=0}^{q-1} w_{k+lq}}^{M,qm} \bigoplus_{l=0}^{q-1} A_{k+lq}\, \mathrm{G}_{[-]_{l=0}^{q-1} r_{k+lq}}^{N,qn}.$$

Corollary 8.15 (Gather/Scattter Splitting)**.** A given iterative sum

$$A = \sum_{j=0}^{p-1} \mathrm{S}_{w_j}^{M,m_j}\, A_j\, \mathrm{G}_{r_j}^{N,n_j} \quad \text{with} \quad A_j \in \mathbb{C}^{m_j \times n_j}$$

is partially unrolled with respect to a partition of the p iterations

$$\{0, \ldots, p-1\} = \bigcup_{k=0}^{q-1} J_k \quad \text{with} \quad J_i \cap J_j = \varnothing \text{ for } i \neq j$$

by

$$A = \sum_{k=0}^{q-1} \mathrm{S}^{M, \sum_{j \in J_k} m_j}_{[-]_{j \in J_k} w_j} \left(\sum_{j \in J_k} \mathrm{S}^{\sum_{j \in J_k} m_j, m_j}_{i \mapsto i + \sum_{\substack{r \in J_k \\ r < j}} m_r} A_j \, \mathrm{G}^{\sum_{j \in J_k} n_j, n_j}_{i \mapsto i + \sum_{\substack{r \in J_k \\ r < j}} n_r} \right) \mathrm{G}^{N, \sum_{j \in J_k} n_j}_{[-]_{j \in J_k} r_j} .$$

Corollary 8.16 (Gather/Scatter Splitting for Fixed Matrix Size)**.** A given iterative sum

$$A = \sum_{j=0}^{p-1} \mathrm{S}^{M,m}_{w_j} A_j \, \mathrm{G}^{N,n}_{r_j} \quad \text{with} \quad A_j \in \mathbb{C}^{m \times n}$$

is partially unrolled with respect to a partition of the p iterations

$$\{0, \ldots, p-1\} = \bigcup_{k=0}^{q-1} J_k \quad \text{with} \quad J_i \cap J_j = \varnothing \text{ for } i \neq j$$

by

$$A = \sum_{k=0}^{q-1} \mathrm{S}^{M, |J_k| m}_{[-]_{j \in J_k} w_j} \left(\sum_{j \in J_k} \mathrm{S}^{|J_k| m, m}_{i \mapsto i + jm} A_j \, \mathrm{G}^{|J_k| n, n}_{i \mapsto i + jn} \right) \mathrm{G}^{N, |J_k| n}_{[-]_{j \in J_k} r_j} .$$

Corollary 8.17 (mn Gather/Scatter Splitting)**.** If

$$p = qr,$$

a given iterative sum

$$A = \sum_{j=0}^{p-1} \mathrm{S}^{M,m}_{w_j} A_j \, \mathrm{G}^{N,n}_{r_j} \quad \text{with} \quad A_j \in \mathbb{C}^{m \times n}$$

is qr split by

$$A = \sum_{k=0}^{q-1} \mathrm{S}^{M, rm}_{[-]_{l=0}^{r-1} w_{kr+l}} \left(\sum_{l=0}^{r-1} \mathrm{S}^{rm, m}_{i \mapsto i + lm} A_{kr+l} \, \mathrm{G}^{rn, n}_{i \mapsto i + ln} \right) \mathrm{G}^{N, rn}_{[-]_{l=0}^{r-1} r_{kr+l}}$$

and rq-fold split by

$$A = \sum_{k=0}^{r-1} \mathrm{S}^{M, qm}_{[-]_{l=0}^{q-1} w_{k+lq}} \left(\sum_{l=0}^{q-1} \mathrm{S}^{qm, m}_{i \mapsto i + lm} A_{k+lq} \, \mathrm{G}^{qn, n}_{i \mapsto i + ln} \right) \mathrm{G}^{N, qn}_{[-]_{l=0}^{q-1} r_{k+lq}} .$$

8.4 Fusion of Incompatible Sums

Definition 169 (Fuseable Incompatible Iterative Sums)**.** A product

$$A \, B$$

of two iterative sums

$$A = \sum_{j=0}^{m-1} \mathrm{S}^{N_3, n_{3,j}}_{w_{1,j}} A_j \, \mathrm{G}^{N, n_{2,j}}_{r_{1,j}} \quad \text{with} \quad A_j \in \mathbb{C}^{n_{3,j} \times n_{2,j}}$$

and

$$B = \sum_{j=0}^{n-1} \mathrm{S}^{N, n_{1,j}}_{w_{0,j}} B_j \, \mathrm{G}^{N_0, n_{0,j}}_{r_{0,j}} \quad \text{with} \quad A_j \in \mathbb{C}^{n_{1,j} \times n_{0,j}}$$

are fuseable incompatible, if

$$\bigcup_{j=0}^{m-1} \Lambda(r_{1,j}) = \bigcup_{j=0}^{n-1} \Lambda(w_{0,j})$$

but

$$\Lambda\Big(\{ r_{1,j} \}_{j=0,\ldots,m-1} \Big) \neq \Lambda\Big(\{ w_{0,j} \}_{j=0,\ldots,n-1} \Big).$$

Theorem 8.15 (Partitioning into Compatible Subsets)**.** For two families of index mapping functions

$$\{r_j\}_{j\in M} \quad \text{and} \quad \{w_j\}_{j\in N}$$

with

$$\bigcup_{j\in M} \Lambda(r_j) = \bigcup_{j\in N} \Lambda(w_j)$$

but

$$\Lambda\Big(\{r_j\}_{j\in M}\Big) \neq \Lambda\Big(\{w_j\}_{j\in N}\Big)$$

partitions

$$M = \bigcup_{j=0}^{k-1} M_j \quad \text{and} \quad N = \bigcup_{j=0}^{k-1} N_j$$

with minimal

$$|M_j| \quad \text{and} \quad |N_j| \quad \forall j \text{ with } 0 \le j < k$$

exist, that

$$\Lambda\left(\left\{[-]_{i\in M_j} r_i\right\}_{j=0,\dots,k-1}\right) = \Lambda\left(\left\{[-]_{i\in N_j} w_i\right\}_{j=0,\dots,k-1}\right).$$

Corollary 8.18 (Compatibility by Partial Unrolling)**.** A product

$$A\,B$$

of two fuseable incompatible iterative sums

$$A = \sum_{j=0}^{m-1} \mathrm{S}_{w_{1,j}}^{R_3,n_{3,j}}\, A_j\, \mathrm{G}_{r_{1,j}}^{R,n_{2,j}} \quad \text{with} \quad A_j \in \mathbb{C}^{n_{3,j}\times n_{2,j}}$$

and

$$B = \sum_{j=0}^{n-1} \mathrm{S}_{w_{0,j}}^{R,n_{1,j}}\, B_j\, \mathrm{G}_{r_{0,j}}^{R_0,n_{0,j}} \quad \text{with} \quad A_j \in \mathbb{C}^{n_{1,j}\times n_{0,j}}$$

is transformed into a product of two compatible iterative sums by applying Theorem 8.15 with

$$\{r_j\}_{j\in M} := \{r_{1,j}\}_{j=0,\dots,m-1} \quad \text{and} \quad \{w_j\}_{j\in N} := \{w_{0,j}\}_{j=0,\dots,n-1}$$

and using Corollary 8.12 leading to

$$A = \sum_{i=0}^{k-1} \mathrm{S}_{[-]_{j\in M_i} w_{1,j}}^{R_3,\sum_{j\in M_i} n_{3,j}} \bigoplus_{j\in M_i} A_j\, \mathrm{G}_{[-]_{j\in M_i} r_{1,j}}^{R,\sum_{j\in M_i} n_{2,j}}$$

and

$$B = \sum_{i=0}^{k-1} \mathrm{S}_{[-]_{j\in N_i} w_{0,j}}^{R,\sum_{j\in N_i} n_{1,j}} \bigoplus_{j\in N_i} B_j\, \mathrm{G}_{[-]_{j\in N_i} r_{0,j}}^{R_0,\sum_{j\in N_i} n_{0,j}}.$$

Corollary 8.19 (Compatibility by Iteration Splitting)**.** A product

$$A\,B$$

of two fuseable incompatible iterative sums

$$A = \sum_{j=0}^{m-1} \mathrm{S}_{w_{1,j}}^{R_3,n_{3,j}}\, A_j\, \mathrm{G}_{r_{1,j}}^{R,n_{2,j}} \quad \text{with} \quad A_j \in \mathbb{C}^{n_{3,j}\times n_{2,j}}$$

and

$$B = \sum_{j=0}^{n-1} \mathrm{S}_{w_{0,j}}^{R,n_{1,j}}\, B_j\, \mathrm{G}_{r_{0,j}}^{R_0,n_{0,j}} \quad \text{with} \quad A_j \in \mathbb{C}^{n_{1,j}\times n_{0,j}}$$

is transformed into a product of two compatible iterative sums by applying Theorem 8.15 with

$$\{r_j\}_{j\in M} := \{r_{1,j}\}_{j=0,\dots,m-1} \quad \text{and} \quad \{w_j\}_{j\in N} := \{w_{0,j}\}_{j=0,\dots,n-1}$$

and using Corollary 8.15 leading to

$$A = \sum_{i=0}^{k-1} \mathrm{S}_{[-]_{j\in M_i} w_{1,j}}^{R_3,\sum_{j\in M_i} n_{3,j}} \left(\sum_{j\in M_i} \mathrm{S}_{i\mapsto i+\sum_{\substack{r\in M_i\\ r<j}} n_{3,r}}^{\sum_{j\in M_i} n_{3,j},n_{3,j}}\, A_j\, \mathrm{G}_{i\mapsto i+\sum_{\substack{r\in M_i\\ r<j}} n_{2,r}}^{\sum_{j\in M_i} n_{2,j},n_{2,j}} \right) \mathrm{G}_{[-]_{j\in M_i} r_{1,j}}^{R,\sum_{j\in M_i} n_{2,j}}$$

and

$$B = \sum_{i=0}^{k-1} \mathrm{S}_{[-]_{j\in N_i} w_{0,j}}^{R,\sum_{j\in N_i} n_{1,j}} \left(\sum_{j\in N_i} \mathrm{S}_{i\mapsto i+\sum_{\substack{r\in N_i\\ r<j}} n_{1,r}}^{\sum_{j\in N_i} n_{1,j},n_{1,j}}\, B_j\, \mathrm{G}_{i\mapsto i+\sum_{\substack{r\in N_i\\ r<j}} n_{0,r}}^{\sum_{j\in N_i} n_{0,j},n_{0,j}} \right) \mathrm{G}_{[-]_{j\in N_i} r_{0,j}}^{R_0,\sum_{j\in N_i} n_{0,j}}.$$

Corollary 8.20 (Compatibility for Fixed Matrix Size by Partial Unrolling)**.** A product

$$A\,B$$

of two fuseable incompatible iterative sums

$$A = \sum_{j=0}^{m-1} \mathrm{S}_{w_{1,j}}^{R_3,m_1}\, A_j\, \mathrm{G}_{r_{1,j}}^{R,n_1} \quad \text{with} \quad A_j \in \mathbb{C}^{m_1\times n_1}$$

and

$$B = \sum_{j=0}^{n-1} \mathrm{S}_{w_{0,j}}^{R,m_0}\, B_j\, \mathrm{G}_{r_{0,j}}^{R_0,n_0} \quad \text{with} \quad A_j \in \mathbb{C}^{m_0\times n_0}$$

is transformed into a product of two compatible iterative sums by applying Theorem 8.15 with

$$\{r_j\}_{j\in M} := \{r_{1,j}\}_{j=0,\dots,m-1} \quad \text{and} \quad \{w_j\}_{j\in N} := \{w_{0,j}\}_{j=0,\dots,n-1}$$

and using Corollary 8.12 leading to

$$A = \sum_{i=0}^{k-1} \mathrm{S}_{[-]_{j\in M_i} w_{1,j}}^{R_3,|M_i|m_1} \bigoplus_{j\in M_i} A_j\, \mathrm{G}_{[-]_{j\in M_i} r_{1,j}}^{R,|M_i|n_1}$$

and

$$B = \sum_{i=0}^{k-1} \mathrm{S}_{[-]_{j\in N_i} w_{0,j}}^{R,|N_i|m_0} \bigoplus_{j\in N_i} B_j\, \mathrm{G}_{[-]_{j\in N_i} r_{0,j}}^{R_0,|N_i|n_0}.$$

Corollary 8.21 (Compatibility for Fixed Matrix Size by Iteration Splitting)**.** A product

$$A\,B$$

of two fuseable incompatible iterative sums

$$A = \sum_{j=0}^{m-1} \mathrm{S}_{w_{1,j}}^{R_3,m_1}\, A_j\, \mathrm{G}_{r_{1,j}}^{R,n_1} \quad \text{with} \quad A_j \in \mathbb{C}^{m_1\times n_1}$$

and

$$B = \sum_{j=0}^{n-1} \mathrm{S}_{w_{0,j}}^{R,m_0}\, B_j\, \mathrm{G}_{r_{0,j}}^{R_0,n_0} \quad \text{with} \quad A_j \in \mathbb{C}^{m_0\times n_0}$$

is transformed into a product of two compatible iterative sums by applying Theorem 8.15 with

$$\{r_j\}_{j\in M} := \{r_{1,j}\}_{j=0,\dots,m-1} \quad \text{and} \quad \{w_j\}_{j\in N} := \{w_{0,j}\}_{j=0,\dots,n-1}$$

and using Corollary 8.15 leading to

$$A = \sum_{t=0}^{k-1} \mathrm{S}^{R_3,|M_t|m_1}_{[-]_{j\in M_t} w_{1,j}} \left(\sum_{j\in M_t} \mathrm{S}^{|M_t|m_1,m_1}_{i\mapsto i+jm_1}\, A_j\, \mathrm{G}^{|M_t|n_1,n_1}_{i\mapsto i+jn_1} \right) \mathrm{G}^{R,|M_t|n_1}_{[-]_{j\in M_t} r_{1,j}}$$

and

$$B = \sum_{t=0}^{k-1} \mathrm{S}^{R,|N_t|m_0}_{[-]_{j\in N_t} w_{0,j}} \left(\sum_{j\in N_t} \mathrm{S}^{|N_t|m_0,m_0}_{i\mapsto i+jm_0}\, B_j\, \mathrm{G}^{|N_t|n_0,n_0}_{i\mapsto i+jn_0} \right) \mathrm{G}^{R_0,|N_t|n_0}_{[-]_{j\in N_t} r_{0,j}}\,.$$

Example 8.5 (Fusion of Incompatible Loops)**.** The construct

$$A = (\mathrm{DFT}_3 \otimes \mathrm{I}_{2r}) \bigoplus_{j=0}^{3r-1} \mathrm{R}_{\alpha_i} \tag{18}$$

should be implemented as a single loop. Theorem 7.8 provides

$$\mathrm{DFT}_3 \otimes \mathrm{I}_{2r} = \sum_{j=0}^{2r-1} \mathrm{S}^{6r,3}_{i\mapsto j+2ri}\, \mathrm{DFT}_3\, \mathrm{G}^{6r,3}_{i\mapsto j+2ri}$$

and Theorem 7.2 provides

$$\bigoplus_{j=0}^{3r-1} \mathrm{R}_{\alpha_i} = \sum_{j=0}^{3r-1} \mathrm{S}^{6r,2}_{i\mapsto 2j+i}\, \mathrm{R}_{\alpha_i}\, \mathrm{G}^{6r,2}_{i\mapsto 2j+i}\,.$$

The definition of

$$r_j : \begin{cases} \{0,1,2\} \to \{0,\dots,6r-1\} \\ i \mapsto j+2ri \end{cases} \qquad \text{for} \quad 0 \le j < 2r$$

and

$$w_j : \begin{cases} \{0,1\} \to \{0,\dots,6rj\} \\ i \mapsto 2j+i \end{cases} \qquad \text{for} \quad 0 \le j < 3r$$

leads to

$$\bigcup_{j=0}^{2r-1} \Lambda(r_j) = \{0,\dots,6r-1\}$$

and

$$\bigcup_{j=0}^{3r-1} \Lambda(w_j) = \{0,\dots,6r-1\}$$

but from the definition of

$$\{r_j\}_{j=0,\dots,2r-1} \quad \text{and} \quad \{w_j\}_{j=0,\dots,3r-1}$$

and Lemma 6.2 follows that

$$\Lambda\Big(\{r_j\}_{j=0,\dots,2r-1}\Big) \neq \Lambda\Big(\{w_j\}_{j=0,\dots,3r-1}\Big).$$

Thus, the product is fuseable incompatible. Definition of

$$M = \{0,\dots,2r-1\} \quad \text{for} \quad N = \{0,\dots,3r-1\}$$

and application of Theorem 8.15 with $\{r_j\}_{j\in M}$ and $\{w_j\}_{j\in N}$ leads to the partitions

$$M_k := \{2k, 2k+1\} \quad \text{for} \quad 0 \le k < r$$

as well as

$$N_k := \{k, k+r, k+2r\} \quad \text{for} \quad 0 \le k < r.$$

Definition of

$$r'_{1,k} := \mathop{[\text{—}]}\limits_{j \,\in\, M_k} r_{1,j} \quad \text{for} \quad 0 \le k < r$$

with

$$|M_k| = 2 \quad \text{for any} \quad j : 0 \le j < r.$$

leads to

$$r'_{1,k} : \begin{cases} \{0,\dots,5\} \to \{0,\dots,6r-1\} \\ i \mapsto \begin{cases} 2k + 2ri & \text{for } 0 \le i < 3 \\ 2k+1+2r(i-3) & \text{for } 3 \le i < 6 \end{cases} \end{cases} \quad \text{with} \quad 0 \le k < r$$

and further simplification leads to

$$r'_{1,k} : \begin{cases} \{0,\dots,5\} \to \{0,\dots,6r-1\} \\ i \mapsto 2k + \left\lfloor \frac{i}{3} \right\rfloor + 2r(i \bmod 3) \end{cases} \quad \text{with} \quad 0 \le k < r.$$

The same operation leads to

$$w'_{1,k} := \mathop{[\text{—}]}\limits_{j \,\in\, M_k} w_{1,j} \quad \text{for} \quad 0 \le k < r$$

with

$$w'_{1,k} : \begin{cases} \{0,\dots,5\} \to \{0,\dots,6r-1\} \\ i \mapsto 2k + \left\lfloor \frac{i}{3} \right\rfloor + 2r(i \bmod 3) \end{cases} \quad \text{with} \quad 0 \le k < r.$$

Definition of

$$r'_{0,k} := \mathop{[\text{—}]}\limits_{j \,\in\, N_k} r_{0,j} \quad \text{with} \quad 0 \le k < r$$

with

$$|N_k| = 3 \quad \text{for any} \quad j : 0 \le j < r$$

leads to

$$r'_{0,k} : \begin{cases} \{0,\dots,5\} \to \{0,\dots,6r-1\} \\ i \mapsto \begin{cases} 2k + i & \text{for } 0 \le i < 2 \\ 2k + 2r + (i-2) & \text{for } 2 \le i < 4 \\ 2k + 4r + (i-4) & \text{for } 4 \le i < 6 \end{cases} \end{cases} \quad \text{with} \quad 0 \le k < r$$

and further simplification leads to

$$r'_{0,k} : \begin{cases} \{0,\dots,5\} \to \{0,\dots,6r-1\} \\ i \mapsto 2k + 2r \left\lfloor \frac{i}{2} \right\rfloor + (i \bmod 2) \end{cases} \quad \text{with} \quad 0 \le k < r.$$

The same operation leads to

$$w'_{0,k} := \mathop{[\text{—}]}\limits_{j \,\in\, N_k} w_{0,j} \quad \text{with} \quad 0 \le k < r$$

with

$$w'_{0,k} : \begin{cases} \{0,\dots,5\} \to \{0,\dots,6r-1\} \\ i \mapsto 2k + 2r\left\lfloor \frac{i}{2} \right\rfloor + (i \bmod 2) \end{cases} \qquad \text{with} \quad 0 \le k < r.$$

This provides loop compatibility by

$$\Lambda\left(\left\{r'_{1,j}\right\}_{j=0,\dots,r-1}\right) = \Lambda\left(\left\{w'_{0,j}\right\}_{j=0,\dots,r-1}\right).$$

Now the loops can be fused. Theorem 8.2 must be applied as

$$\Lambda\left(r'_{1,j}\right) = \Lambda\left(w'_{0,j}\right) \quad \text{for} \quad 0 \le j < r$$

but

$$r'_{1,j} \ne w'_{0,j} \quad \text{for} \quad 0 \le j < r.$$

$$\pi_j = {w'_{0,j}}^{-1} \circ r'_{1,j}.$$

The pseudo inverse of $w'_{0,j}$ is given by

$${w'_{0,j}}^{-1} : \begin{cases} \{2j, 2j+1, 2r+2j, 2r+2j+1, 4r+2j, 4r+2j+1\} \to \{0,\dots,5\} \\ i \mapsto (i \bmod 2) + 2\left\lfloor \frac{i}{2r} \right\rfloor \end{cases}$$

and

$$\pi_j = {w'_{0,j}}^{-1} \circ r'_{1,j}$$

leading to

$$\pi_j : \begin{cases} \{0,\dots,5\} \to \{0,\dots,6r-1\} \\ i \mapsto \left(\left(2j + \left\lfloor \frac{i}{3} \right\rfloor + 2r(i \bmod 3)\right) \bmod 2\right) + 2\left\lfloor \frac{2j+\left\lfloor \frac{i}{3} \right\rfloor + 2r(i \bmod 3)}{2r} \right\rfloor \end{cases}$$

which simplifies to

$$\pi : \begin{cases} \{0,\dots,5\} \to \{0,\dots,6r-1\} \\ i \mapsto \left\lfloor \frac{i}{3} \right\rfloor + 2(i \bmod 3) \end{cases}.$$

In addition,

$$\pi = \ell_2^6$$

finally leading to

$$A = \sum_{j=0}^{r-1} \mathrm{S}^{6r-1,6}_{i \mapsto 2j+\left\lfloor \frac{i}{3} \right\rfloor + 2r(i \bmod 3)} (\mathrm{I}_2 \otimes \mathrm{DFT}_3)\, \mathrm{L}_2^6 \bigoplus_{k=0}^{3} \mathrm{R}_{\alpha_{3j+k}}\, \mathrm{G}^{6r-1,6}_{i \mapsto 2j+2r\left\lfloor \frac{i}{2} \right\rfloor + (i \bmod 2)}.$$

Note, that the simple formula manipulation

$$(\mathrm{DFT}_3 \otimes \mathrm{I}_{2r}) \bigoplus_{j=0}^{3r-1} \mathrm{R}_{\alpha_i} = \left((\mathrm{I}_2 \otimes \mathrm{DFT}_3)^{\mathrm{L}_2^6} \otimes \mathrm{I}_r\right) \bigoplus_{j=0}^{r-1} \left(\mathrm{R}_{\alpha_{3j}} \oplus \mathrm{R}_{\alpha_{3j+1}} \oplus \mathrm{R}_{\alpha_{3j+2}}\right)$$

would immediately produce compatible iterative sums.

9 h-Separability of Permutations

Definition 170 (Index Mapping Functions)**.** Index mapping functions are of form

$$f : \begin{cases} \{i_0, ..., i_m\} \to \{j_0, ..., j_n\} \\ i \mapsto f(i) \end{cases} \qquad \text{with} \quad \begin{array}{l} i_k, j_l \in \mathbb{N}, \\ 0 \le k \le m, \\ 0 \le l \le n, \\ m \le n \end{array} .$$

Definition 171 (Concatenation of Index Mapping Functions)**.** The concatenation of the two index mapping functions

$$f : \begin{cases} I \to J \\ i \mapsto f(i) \end{cases} \qquad \text{and} \quad g : \begin{cases} J \to K \\ i \mapsto g(i) \end{cases}$$

is given by the index mapping function

$$g \circ f : \begin{cases} I \to K \\ i \mapsto g(f(i)). \end{cases}$$

Definition 172 (Cross-Product of Index Mapping Functions)**.** The cross product of the two index mapping functions

$$f : \begin{cases} I \to J \\ i \mapsto f(i) \end{cases} \qquad \text{and} \quad g : \begin{cases} K \to L \\ j \mapsto g(j) \end{cases}$$

is given by the index mapping function

$$f \times g : \begin{cases} I \times K \to J \times L \\ (i, j) \mapsto \big(f(i), g(j)\big) \end{cases} .$$

Definition 173 (Restriction of Index Mapping Functions)**.** For an index mapping function f with

$$f : \begin{cases} I \to J \\ i \mapsto f(i) \end{cases} ,$$

the restriction of f to $I_1 \subseteq I$ is defined by

$$f|_{I_1} : \begin{cases} I_1 \to J \\ i \mapsto f(i) \end{cases} .$$

Definition 174 (Pseudo Inversion of Index Mapping Functions)**.** For an injective index mapping function f with

$$f : \begin{cases} I \to J \\ i \mapsto f(i) \end{cases} ,$$

the pseudo inverse f^{-1} is defined by

$$f^{-1} : \begin{cases} f(I) \to I \\ i \mapsto j \quad \text{with } f(j) = i \end{cases}$$

Definition 175 (Fusion of Index Mapping Functions)**.** The fusion of two index mapping functions

$$f : \begin{cases} I \to J \\ i \mapsto f(i) \end{cases} \qquad \text{and} \quad g : \begin{cases} K \to L \\ i \mapsto g(i) \end{cases} \qquad \text{with} \quad I \cap K = \varnothing$$

is given by the generating function

$$f \cup g : \begin{cases} I \cup K \to J \cup L \\ i \mapsto \begin{cases} f(i) & \text{if } i \in I \\ g(i) & \text{if } i \in K \end{cases} \end{cases} .$$

Definition 176 (Interval Mapping Function)**.** A index mapping function of form

$$f : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, N-1\} \\ i \mapsto f(i) \end{cases}$$

is called interval mapping function.

Definition 177 (h-Separability of Interval Mapping Functions)**.** A family of interval mapping functions

$$\{f_j\}_{j=0,\ldots,m-1}\,, \quad \text{with} \quad f_j : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, N-1\} \\ i \mapsto f'(j,i) \end{cases}$$

with

$$f' : \begin{cases} \{0, \ldots, m-1\} \times \{0, \ldots, n-1\} \to \{0, \ldots, N-1\} \\ i \mapsto f'(j,i) \end{cases}$$

is additive h-separable, if for the function

$$h : \begin{cases} \{0, \ldots, N-1\} \times \{0, \ldots, N-1\} \to \{0, \ldots, N-1\} \\ (i,j) \mapsto h(i,j) \end{cases}$$

functions

$$b : \begin{cases} \{0, \ldots, m-1\} \to \{0, \ldots, N-1\} \\ j \mapsto b(j) \end{cases}$$

and

$$s : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, N-1\} \\ i \mapsto s(j) \end{cases}$$

exist, such that

$$f' = h \circ (b \times s).$$

Definition 178 (Permutation Generating Function)**.** Permutation generating functions are bijective interval mapping functions of type

$$\pi : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, n-1\} \\ i \mapsto \pi(i) \end{cases}$$

with the permutation

$$\pi \in \mathrm{S}_n\,.$$

Definition 179 (h-Separability of Permutations w.r.t. Functions)**.** A permutation generation function

$$\pi : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, n-1\} \\ i \mapsto \pi(i) \end{cases} \qquad \text{with} \quad \pi \in \mathrm{S}_n$$

is h-separable with respect to a family of interval mapping functions

$$\{f_j\}_{j=0,\ldots,m-1} \quad \text{with} \quad f_j : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, N-1\} \\ i \mapsto f_j(i) \end{cases}$$

if the family of interval mapping functions

$$\{f_j \circ \pi\}_{j=0,\ldots,m-1}$$

is h-separable.

Definition 180 (h-Inverse-Separability of Permutations w.r.t. Functions)**.** A permutation generation function

$$\pi : \begin{cases} \{0, \dots, n-1\} \to \{0, \dots, n-1\} \\ i \mapsto \pi(i) \end{cases} \qquad \text{with} \quad \pi \in \mathrm{S}_n$$

is h-inverse-separable with respect to a family of interval mapping functions

$$\{f_j\}_{j=0,\dots,m-1} \quad \text{with} \quad f_j : \begin{cases} \{0, \dots, n-1\} \to \{0, \dots, N-1\} \\ i \mapsto f_j(i) \end{cases}$$

if the family of interval mapping functions

$$\left\{\pi^{-1} \circ f_j\right\}_{j=0,\dots,m-1}$$

is h-separable.

Definition 181 (h-Separability of Permutations)**.** A permutation generation function

$$\pi : \begin{cases} \{0, \dots, n-1\} \to \{0, \dots, n-1\} \\ i \mapsto \pi(i) \end{cases} \qquad \text{with} \quad \pi \in \mathrm{S}_n$$

is h-separable if it can be decomposed into two h-separable families of interval mapping functions

$$\left\{r_j\right\}_{j=0,\dots,k-1} \quad \text{and} \quad \left\{w_j\right\}_{j=0,\dots,k-1}$$

providing

$$\pi = \bigcup_{j=0}^{k-1} r_j \circ w_j^{-1}.$$

Definition 182 (Stride Permutation Generating Function)**.** The stride permutation

$$\mathrm{L}_m^{mn} = \mathrm{perm}\left(\ell_m^{mn}\right)$$

is generated by

$$\ell_m^{mn} : \begin{cases} \{0, \dots, mn-1\} \to \{0, \dots, mn-1\} \\ i \mapsto \begin{cases} (im) \bmod (mn-1) & \text{if } i < mn-1 \\ mn-1 & \text{if } i = mn-1 \end{cases} \end{cases}.$$

Lemma 9.1. The stride permutation

$$\mathrm{L}_m^{mn} = \mathrm{perm}\left(\ell_m^{mn}\right)$$

is generated by

$$\ell_m^{mn} : \begin{cases} \{0, \dots, mn-1\} \to \{0, \dots, mn-1\} \\ i \mapsto \left\lfloor \frac{i}{n} \right\rfloor + m(i \bmod n) \end{cases}.$$

Lemma 9.2. The stride permutation generating function ℓ_m^{mn} can be decomposed into

$$\ell_m^{mn} = \bigcup_{j=0}^{m-1} r_j \circ w_j^{-1}$$

with

$$r_j : \begin{cases} \{0, \dots, n-1\} \to \{0, \dots, mn-1\} \\ i \mapsto j + im \end{cases}$$

and

$$w_j : \begin{cases} \{0, \dots, n-1\} \to \{0, \dots, mn-1\} \\ i \mapsto jn + i \end{cases}.$$

Lemma 9.3. The stride permutation generating function ℓ_m^{mn} can be decomposed into

$$\ell_m^{mn} = \bigcup_{j=0}^{n-1} r_j \circ w_j^{-1}$$

with

$$r_j : \begin{cases} \{0, \ldots, m-1\} \to \{0, \ldots, mn-1\} \\ i \mapsto jm + i \end{cases}$$

and

$$w_j : \begin{cases} \{0, \ldots, m-1\} \to \{0, \ldots, mn-1\} \\ i \mapsto j + in \end{cases} \qquad .$$

10 Overview

10.1 Definitions

Definition 183 (Matrix Generating Function with One Parameter)**.** Matrix generating functions with one parameter are of type

$$f : \begin{cases} \{0, \ldots, n-1\} \to \mathbb{C}^{M \times N} \\ i \mapsto [f_{j,k}(i)]_{\substack{j=0,\ldots,M-1 \\ k=0,\ldots,N-1}} \end{cases} .$$

Definition 184 (Permutation Generating Function)**.** Permutation generating functions are of type

$$\pi : \begin{cases} \{0, \ldots, n-1\} \to \{0, \ldots, n-1\} \\ i \mapsto \pi(i) \end{cases}$$

with the permutation

$$\pi \in \mathrm{S}_n .$$

10.2 Integer Identities

10.3 Index Function Operators

10.4 Formula Identities

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Definition 185 (Affine Basis Function)**.**

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Definition 186 (2D Combine Function).

$$\mathrm{h}_{\left[\begin{smallmatrix} b \\ s \end{smallmatrix}\right]}^{n\to N} : \begin{cases} \mathbb{I}_n \to \mathbb{I}_N \\ i \mapsto b + is \end{cases}$$

Definition 187 (2D Combine Function Recursion).

$$\mathrm{h}_{\left[\begin{smallmatrix} b \\ s \end{smallmatrix}\right]}^{n\to N} : \begin{cases} \mathbb{I}_n \to \mathbb{I}_N \\ i \mapsto \begin{cases} b & \text{if } i = 0 \\ \mathrm{h}_{\left[\begin{smallmatrix} b \\ s \end{smallmatrix}\right]}^{n\to N}(i-1) + s & \text{else} \end{cases} \end{cases}$$

Definition 188 (Parameter Recursion Function).

$$\mathrm{s}_{j,n} : \begin{cases} \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N} \\ \left[\begin{smallmatrix} b \\ s \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} b+jn \\ s \end{smallmatrix}\right] \end{cases}$$

Property 12.1 (Parameter Recursion Function).

$$\mathrm{s}_{j,n} = (jn)_{+}^{\mathbb{N}\to\mathbb{N}} \times (1)_{\times}^{\mathbb{N}\to\mathbb{N}}$$

Definition 189 (Parameter Recursion Function).

$$\mathrm{t}_{j,n} : \begin{cases} \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N} \\ \left[\begin{smallmatrix} b \\ s \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} b+j \\ sn \end{smallmatrix}\right] \end{cases}$$

Property 12.2 (Parameter Recursion Function).

$$\mathrm{t}_{j,n} = (j)_{+}^{\mathbb{N}\to\mathbb{N}} \times (n)_{\times}^{\mathbb{N}\to\mathbb{N}}$$

Property 12.3 (Identity Function).

$$\imath_n = \mathrm{h}_{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]}^{n\to n}$$

Identity 12.1.

$$\mathrm{h}_{\vec{p}}^{n\to N} \otimes (j)_m = \mathrm{h}_{\mathrm{t}_{j,m}(\vec{p})}^{n\to Nm}$$

Identity 12.2.

$$(j)_m \otimes \mathrm{h}_{\vec{p}}^{n\to N} = \mathrm{h}_{\mathrm{s}_{j,n}(\vec{p})}^{n\to Nm}$$

Identity 12.3.

$$\mathrm{h}_{\vec{p}}^{mn\to N} \circ \mathrm{h}_{\mathrm{s}_{j,n}\left(\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\right)}^{m\to mn} = \mathrm{h}_{\mathrm{s}_{j,n}(\vec{p})}^{n\to N}$$

Identity 12.4.

$$\mathrm{h}_{\vec{p}}^{mn\to N} \circ \mathrm{h}_{\mathrm{t}_{j,n}\left(\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\right)}^{m\to mn} = \mathrm{h}_{\mathrm{t}_{j,n}(\vec{p})}^{n\to N}$$

13 Examples

13.1 DFT Examples

Example 13.1 (Pulling in the Stride Permutation)**.**

$$\left(\mathrm{I}_m \otimes A^{n\times n}\right) \mathrm{L}_m^{mn}$$

Step 1: SPL → Σ-SPL.

$$= \left(\sum_{j=0}^{m-1} \mathrm{S}_{(j)_m \otimes \imath_n} A^{n\times n}\, \mathrm{G}_{(j)_m \otimes \imath_n}\right) \mathrm{perm}\left(\ell_m^{mn}\right)$$

Step 2: Pull permutation into iterative sum.

$$= \sum_{j=0}^{m-1} \mathrm{S}_{(j)_m \otimes \imath_n} A^{n\times n}\, \mathrm{G}_{(j)_m \otimes \imath_n} \mathrm{perm}\left(\ell_m^{mn}\right)$$

Step 3: Pull permutation into gather matrix.

$$= \sum_{j=0}^{m-1} \mathrm{S}_{(j)_m \otimes \imath_n} A^{n\times n}\, \mathrm{G}_{\ell_m^{mn} \circ \left((j)_m \otimes \imath_n\right)}$$

Step 4: Flip function tensor product.

$$= \sum_{j=0}^{m-1} \mathrm{S}_{(j)_m \otimes \imath_n} A^{n\times n}\, \mathrm{G}_{\imath_n \otimes (j)_m}$$

Example 13.2 (Permutations in two Cooley-Tukey Recursion Steps)**.**

$$\mathrm{I}_k \otimes \Big(\left(\mathrm{I}_m \otimes A^{n\times n}\right) \mathrm{L}_m^{mn} \Big) \mathrm{L}_k^{kmn}$$

Step 1: SPL → Σ-SPL.

$$= \left(\sum_{i=0}^{k-1} \mathrm{S}_{(i)_k \otimes \imath_{mn}} \left(\sum_{j=0}^{m-1} \mathrm{S}_{(j)_m \otimes \imath_n} A^{n\times n}\, \mathrm{G}_{(j)_m \otimes \imath_n}\right) \mathrm{perm}\left(\ell_m^{mn}\right) \mathrm{G}_{(i)_k \otimes \imath_{mn}}\right) \mathrm{perm}\left(\ell_k^{kmn}\right)$$

Step 2: Pull permutation, gather and scatter into iterative sum.

$$= \sum_{i=0}^{k-1} \sum_{j=0}^{m-1} \mathrm{S}_{(i)_k \otimes \imath_{mn}} \mathrm{S}_{(j)_m \otimes \imath_n} A^{n\times n}\, \mathrm{G}_{(j)_m \otimes \imath_n} \mathrm{perm}\left(\ell_m^{mn}\right) \mathrm{G}_{(i)_k \otimes \imath_{mn}} \mathrm{perm}\left(\ell_k^{kmn}\right)$$

Step 3: Fuse permutations, gather and scatter.

$$= \sum_{i=0}^{k-1} \sum_{j=0}^{m-1} \mathrm{S}_{\left((i)_k \otimes \imath_{mn}\right) \circ \left((j)_m \otimes \imath_n\right)} A^{n\times n}\, \mathrm{G}_{\ell_k^{kmn} \circ \left((i)_k \otimes \imath_{mn}\right) \circ \ell_m^{mn} \circ \left((j)_m \otimes \imath_n\right)}$$

Step 4: Flip function tensor products.

$$= \sum_{i=0}^{k-1} \sum_{j=0}^{m-1} \mathrm{S}_{\left((i)_k \otimes \imath_{mn}\right) \circ \left((j)_m \otimes \imath_n\right)} A^{n\times n}\, \mathrm{G}_{\left(\imath_{mn} \otimes (i)_k\right) \circ \left(\imath_n \otimes (j)_m\right)}$$

Step 5: Plug in functions.

$$= \sum_{i=0}^{k-1} \sum_{j=0}^{m-1} \mathrm{S}_{(i)_k \otimes (j)_m \otimes \imath_n} A^{n\times n}\, \mathrm{G}_{\imath_n \otimes (j)_m \otimes (i)_k}$$

Example 13.3 (Pull in Dimensionless FFT Permutation)**.**

Example 13.4 (Pulling in a Diagonal Matrix)**.**

$$\left(A^{m\times m} \otimes \mathrm{I}_n\right) D$$

Step 1: SPL $\rightarrow$ Σ-SPL.

$$= \left(\sum_{j=0}^{n-1} \mathrm{S}_{\imath_m \otimes (j)_n}\, A^{m\times m}\, \mathrm{G}_{\imath_m \otimes (j)_n}\right) \mathrm{diag}\left(f^{mn\rightarrow\mathbb{C}}\right) \quad , \quad D \text{ generated by } f^{mn\rightarrow\mathbb{C}}$$

Step 2: Pull diagonal into iterative sum.

$$= \sum_{j=0}^{n-1} \mathrm{S}_{\imath_m \otimes (j)_n}\, A^{m\times m}\, \mathrm{G}_{\imath_m \otimes (j)_n}\, \mathrm{diag}\left(f^{mn\rightarrow\mathbb{C}}\right)$$

Step 3: Commute diagonal and gather matrix.

$$= \sum_{j=0}^{n-1} \mathrm{S}_{\imath_m \otimes (j)_n}\, A^{m\times m}\, \mathrm{diag}\left(f^{mn\rightarrow\mathbb{C}} \circ \left(\imath_m \otimes (j)_n\right)\right) \mathrm{G}_{\imath_m \otimes (j)_n}$$

Example 13.5 (1D DFT Cooley-Tukey Recursion)**.** Applying example 13.8 and 13.4 to

$$\mathrm{DFT}_{mn} = (\mathrm{DFT}_m \otimes \mathrm{I}_n)\, \mathrm{T}_n^{mn}\, (\mathrm{I}_m \otimes \mathrm{DFT}_n)\, \mathrm{L}_n^{mn}$$

leads to the 1D Cooley-Tukey DFT rule in Σ-SPL notation.

$$\mathrm{DFT}_{mn} = \sum_{j=0}^{n-1} \mathrm{S}_{\imath_m \otimes (j)_n}\, \mathrm{DFT}_m\, \mathrm{diag}\left(\mathrm{t}_m^{mn} \circ \left(\imath_m \otimes (j)_n\right)\right) \mathrm{G}_{\imath_m \otimes (j)_n} \sum_{j=0}^{m-1} \mathrm{S}_{(j)_m \otimes \imath_n}\, \mathrm{DFT}_n\, \mathrm{G}_{\imath_n \otimes (j)_m}$$

14 Real DFT

Definition 190 (Packed Real DFT Nonterminals).

$$
\begin{aligned}
\mathrm{RDFT}'_n &:= \mathrm{G}_{(0)_+^{\lfloor n/2\rfloor+1\to n}\otimes\imath_2}\,\overline{\mathrm{DFT}}_n\,\mathrm{S}_{\imath_n\otimes(0)_2}\\
\mathrm{iRDFT}'_n &:= \mathrm{G}_{\imath_n\otimes(0)_2}\,\overline{\mathrm{iDFT}}_n\,\mathrm{diag}\left(\left(\imath^{\lceil n/2\rceil\to\mathbb{R}}\otimes\imath^{2\to\mathbb{R}}\right)\oplus\left(\imath^{\lfloor n/2\rfloor\to\mathbb{R}}\otimes(\pm\imath)^{2\to\mathbb{R}}\right)\right)\mathrm{G}_{\begin{bmatrix}\imath_{\lfloor n/2\rfloor+1}\\ \jmath_{\lceil n/2\rceil-1}\circ(1)_+^{\lceil n/2\rceil-1\to\lfloor n/2\rfloor+1}\end{bmatrix}}\\
\overline{\mathrm{DFT}}'_n &:= D'_n\,\mathrm{G}_{(0)_+^{\lfloor n/2\rfloor+1\to n}\otimes\imath_2}\,\overline{\mathrm{DFT}}_n\quad\text{with}\\
&\qquad D'_n := \begin{cases}\mathrm{diag}\left(\delta_{\mathbb{I}_1}^{2\to\mathbb{R}}\oplus\left(\imath^{n/2-1\to\mathbb{R}}\otimes\imath^{2\to\mathbb{R}}\right)\oplus\delta_{\mathbb{I}_1}^{2\to\mathbb{R}}\right) & n\text{ even}\\ \mathrm{diag}\left(\delta_{\mathbb{I}_1}^{2\to\mathbb{R}}\oplus\left(\imath^{\lfloor n/2\rfloor\to\mathbb{R}}\otimes\imath^{2\to\mathbb{R}}\right)\right) & \text{else}\end{cases}\\
\overline{\mathrm{DFT}}''_n &:= \mathrm{diag}\left(\left(\imath^{\lceil n/2\rceil\to\mathbb{R}}\otimes\imath^{2\to\mathbb{R}}\right)\oplus\left(\imath^{\lfloor n/2\rfloor\to\mathbb{R}}\otimes(\pm\imath)^{2\to\mathbb{R}}\right)\right)\overline{\mathrm{DFT}}_n\\
\overline{\mathrm{DFT}}'''_n &:= D'''_n\,\mathrm{G}_{(0)_+^{\lceil n/2\rceil\to n}\otimes\imath_2}\,\overline{\mathrm{DFT}}_n\quad\text{with}\\
&\qquad D'''_n := \begin{cases}\mathrm{diag}\left(\imath^{n/2\to\mathbb{R}}\otimes\imath^{2\to\mathbb{R}}\right) & n\text{ even}\\ \mathrm{diag}\left(\imath^{\lfloor n/2\rfloor\to\mathbb{R}}\otimes\imath^{2\to\mathbb{R}}\oplus\delta_{\mathbb{I}_1}^{2\to\mathbb{R}}\right) & \text{else}\end{cases}
\end{aligned}
$$

Definition 191 (Twiddle Factors).

$$\overline{\mathrm{T}}'^{mn}_n := \overline{\mathrm{diag}\left(\mathrm{t}_n^{mn}\circ\left(\imath_m\otimes(0)_+^{\lfloor n/2\rfloor+1\to n}\right)\right)}$$

Definition 192 (Output Permutation).

$$r_{m,n,j}^N : \begin{cases}\mathbb{I}_N\to\mathbb{I}_{\lfloor mn/2\rfloor+1}\\ i\mapsto\begin{cases}in+j & \text{if } i<\lceil m/2\rceil\\ mn-in-j & \text{else}\end{cases}\end{cases}$$

$$P_{m,n} := \mathrm{perm}\left(p_{m,n}^{-1}\right)\quad\text{with}\quad p_{m,n} := \begin{cases}\begin{bmatrix}r_{m,n,0}^{\lfloor m/2\rfloor+1}\\ [-]_{j=1}^{n/2}r_{m,n,j}^m\\ r_{m,n,n/2}^{\lceil m/2\rceil}\end{bmatrix} & n\text{ even}\\ \begin{bmatrix}r_{m,n,0}^{\lfloor m/2\rfloor+1}\\ [-]_{j=1}^{\lfloor n/2\rfloor+1}r_{m,n,j}^m\end{bmatrix} & \text{else}\end{cases}$$

Identity 14.1 (Packed RDFT Recursion).

$$\mathrm{RDFT}'_{mn}\to\begin{cases}\left(P_{m,n}\otimes\mathrm{I}_2\right)\left(\overline{\mathrm{DFT}}'_m\oplus\left(\bigoplus_{j=1}^{n/2-1}\overline{\mathrm{DFT}}''_m\right)\oplus\overline{\mathrm{DFT}}'''_m\right)\left(\mathrm{L}_{n/2+1}^{m(n/2+1)}\otimes\mathrm{I}_2\right)\overline{\mathrm{T}}'^{mn}_n\left(\mathrm{I}_m\otimes\mathrm{RDFT}'_n\right)\mathrm{L}_m^{mn} & n\text{ even}\\ \left(P_{m,n}\otimes\mathrm{I}_2\right)\left(\overline{\mathrm{DFT}}'_m\oplus\left(\bigoplus_{j=1}^{\lfloor n/2\rfloor}\overline{\mathrm{DFT}}''_m\right)\right)\left(\mathrm{L}_{\lceil n/2\rceil}^{m(\lceil n/2\rceil)}\otimes\mathrm{I}_2\right)\overline{\mathrm{T}}'^{mn}_n\left(\mathrm{I}_m\otimes\mathrm{RDFT}'_n\right)\mathrm{L}_m^{mn} & \text{else}\end{cases}$$

Identity 14.2 (Pruned RDFT Conquer Step in Sums Notation).

$$\mathrm{RDFT}'_{mn}\to Q_{m,n}\overline{\mathrm{T}}'^{mn}_n\left(\mathrm{I}_m\otimes\mathrm{RDFT}'_n\right)\mathrm{L}_m^{mn}$$

with

$$Q_{m,n} = \begin{cases}\mathrm{S}_{r_{m,n,0}^{\lfloor m/2\rfloor+1}\otimes\imath_2}\,\overline{\mathrm{DFT}}'_m\,\mathrm{G}_{\imath_m\otimes(0)_{n/2+1}\otimes\imath_2}+\sum_{j=1}^{n/2-1}\mathrm{S}_{r_{m,n,j}^m\otimes\imath_2}\,\overline{\mathrm{DFT}}''_m\,\mathrm{G}_{\imath_m\otimes(j)_{n/2+1}\otimes\imath_2}+ & \\ \qquad+\,\mathrm{S}_{r_{m,n,n/2}^{\lceil m/2\rceil}\otimes\imath_2}\,\overline{\mathrm{DFT}}'''_m\,\mathrm{G}_{\imath_m\otimes(n/2)_{n/2+1}\otimes\imath_2} & n\text{ even}\\ \mathrm{S}_{r_{m,n,0}^{\lfloor m/2\rfloor+1}\otimes\imath_2}\,\overline{\mathrm{DFT}}'_m\,\mathrm{G}_{\imath_m\otimes(0)_{n/2+1}\otimes\imath_2}+\sum_{j=1}^{\lfloor n/2\rfloor}\mathrm{S}_{r_{m,n,j}^m\otimes\imath_2}\,\overline{\mathrm{DFT}}''_m\,\mathrm{G}_{\imath_m\otimes(j)_{n/2+1}\otimes\imath_2} & n\text{ odd}\end{cases}$$

Identity 14.3 (Unpruned RDFT Conquer Step in Sums Notation)**.**

$$Q_{m,n} = \sum_{j=1}^{\lfloor n/2 \rfloor} \mathrm{S}_{r^m_{m,n,j} \otimes \imath_2} \, \overline{\mathrm{DFT}}''_m \, \mathrm{G}_{\imath_m \otimes (j)_{n/2+1} \otimes \imath_2}$$

Identity 14.4 (Packed iRDFT Recursion)**.**

$$\mathrm{iRDFT}'_{mn} \rightarrow \mathrm{L}^{mn}_n \left(\mathrm{I}_m \otimes \mathrm{iRDFT}'_n \right) \overline{\mathrm{T}}'^{mn,-1}_n \, \mathrm{L}^{m(\lfloor n/2 \rfloor + 1)}_m \left(\mathrm{I}_{\lfloor n/2 \rfloor + 1} \otimes \overline{\mathrm{iDFT}}_m \right) \mathrm{G}_{[-]^{\lfloor n/2 \rfloor}_{j=0} \, r^m_{m,n,j}}$$

Identity 14.5 (iRDFT Divide Step in Sums Notation)**.**

$$\mathrm{iRDFT}'_{mn} \rightarrow \mathrm{L}^{mn}_n \left(\mathrm{I}_m \otimes \mathrm{iRDFT}'_n \right) \overline{\mathrm{T}}'^{mn,-1}_n Q^{-1}_{m,n}$$

with

$$Q^{-1}_{m,n} = \sum_{j=0}^{\lfloor n/2 \rfloor} \mathrm{S}_{\imath_m \otimes (j)_{\lfloor n/2 \rfloor + 1}} \, \overline{\mathrm{iDFT}}_m \, \mathrm{G}_{r^m_{m,n,j}}$$