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Σ - SPL Example

$$y = (I_2 \otimes F_2) L^4_2 \; x$$

$$(I_2 \otimes F_2) = \begin{pmatrix} 1 & 1 & & \\ 1 & -1 & & \\ & & 1 & 1 \\ & & 1 & -1 \end{pmatrix}$$

$$L^4_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\downarrow$ SPL interpretation

$$y = (I_2 \otimes F_2)(L^4_2 x)$$

$\downarrow$

$$t := L^4_2 x$$
$$y := (I_2 \otimes F_2) t$$

$\downarrow$ Templates

```
for (i = 0; i < 4; i++)
    t[i] = x[i<3 ? (2i mod 3) : 3];
for (i = 0; i < 2; i++) {
    y[2i] = t[2i] + t[2i+1];
    y[2i+1] = t[2i] - t[2i-1];
}
```

Σ-SPL - Example - cont'd

$$(I_2 \otimes F_2) L^4_2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

Template for

$$y = \left((I_2 \otimes F_2) L^4_2\right) x$$

$$\downarrow$$

```
for(i=0; i<2; i++) {
    y[2i] = x[i] + x[i+2];
    y[2i+1] = x[i] - x[i+2];
}
```

Σ-SPL - Idea

$$(I_2 \otimes F_2) = \left(\begin{array}{cc|cc} 1 & 1 & & \\ 1 & -1 & & \\ \hline & & 1 & 1 \\ & & 1 & -1 \end{array}\right) = \left(\begin{array}{cc|cc} 1 & 1 & & \\ 1 & -1 & & \\ \hline & & & \\ & & & \end{array}\right) + \left(\begin{array}{cc|cc} & & & \\ & & & \\ \hline & & 1 & 1 \\ & & 1 & -1 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 1 & 1 & & \\ 1 & -1 & & \\ \hline & & & \\ & & & \end{array}\right) = \underbrace{\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right)}_{S_0} \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{F_2} \underbrace{\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)}_{G_0}$$

$$\left(\begin{array}{cc|cc} & & & \\ & & & \\ \hline & & 1 & 1 \\ & & 1 & -1 \end{array}\right) = \underbrace{\left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right)}_{S_1} \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{F_2} \underbrace{\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)}_{G_1}$$

$$(I_2 \otimes F_2) = \sum_{i=0}^{1} S_i F_2 G_i$$

Σ-SPL - Idea cont'd

$$(I_2 \otimes F_2) L^4_2 = \begin{pmatrix}1&0&1&0\\1&0&-1&0\\0&1&0&1\\0&1&0&-1\end{pmatrix} = \left(\begin{array}{cccc}1&0&1&0\\1&0&-1&0\\\hline &&&\\&&&\end{array}\right) + \left(\begin{array}{cccc}&&&\\&&&\\\hline 0&1&0&1\\0&1&0&-1\end{array}\right)$$

$$\left(\begin{array}{cccc}1&0&1&0\\1&0&-1&0\\\hline &&&\\&&&\end{array}\right) = \underbrace{\left(\begin{array}{cc}1&0\\0&1\\\hline 0&0\\0&0\end{array}\right)}_{S_0} \underbrace{\begin{pmatrix}1&1\\1&-1\end{pmatrix}}_{F_2} \underbrace{\begin{pmatrix}1&0&0&0\\0&0&1&0\end{pmatrix}}_{G_0}$$

$$\left(\begin{array}{cccc}&&&\\&&&\\\hline 0&1&0&1\\0&1&0&-1\end{array}\right) = \underbrace{\left(\begin{array}{cc}0&0\\0&0\\\hline 1&0\\0&1\end{array}\right)}_{S_1} \underbrace{\begin{pmatrix}1&1\\1&-1\end{pmatrix}}_{F_2} \underbrace{\begin{pmatrix}0&1&0&0\\0&0&0&1\end{pmatrix}}_{G_1}$$

$$(I_2 \otimes F_2) L^4_2 = \sum_{i=0}^{1} S_i F_2 G_i$$

Index Mapping Functions

$\mathbb{I}_n = \{0, \dots, n-1\} \supseteq \mathbb{N}_0$ interval

$f : \mathbb{I}_n \to \mathbb{I}_N \; ; \; i \mapsto f(i)$ index mapping function

name, variable, expression, domain, range

Short cut:

$f^{n \to N}$

Examples

$f : \mathbb{I}_2 \to \mathbb{I}_4 \; ; \; i \mapsto 2i$

$0 \to 0$
$1 \to 2$

parameterized functions:

$f_j : \mathbb{I}_n \to \mathbb{I}_N \; ; \; i \mapsto f_j(i)$

Parameter

Examples:

$f_j : \mathbb{I}_2 \to \mathbb{I}_4 \; ; \; i \mapsto 2i + j$

$i =$	0	1
$j=0$	0	2
$j=1$	1	3

$g_j : \mathbb{I}_2 \to \mathbb{I}_4 \; ; \; i \mapsto i + 2j$

$i =$	0	1
$j=0$	0	1
$j=1$	2	3

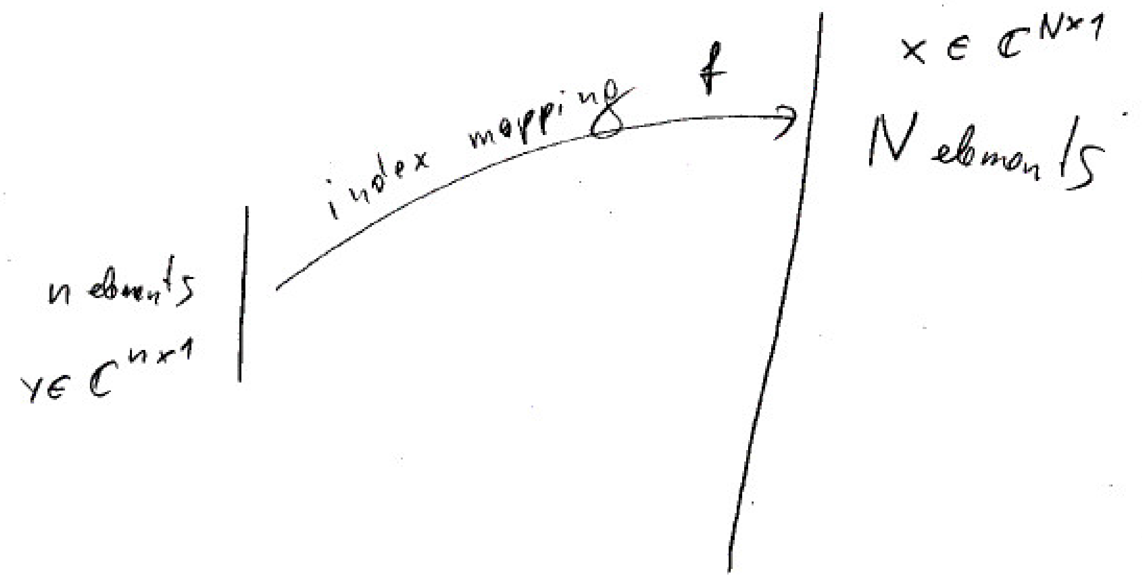
Gather Matrices

Basis vectors:

$$e_i^N = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \\ \\ \\ \leftarrow i\text{th element} \\ \\ \\ \leftarrow \in \mathbb{C}^{N \times 1} \end{matrix}$$

$$\left(e_i^N\right)^T = \left(0 \ldots 0 \,|\, 1 \,|\, 0 \ldots 0 \right) \quad \in \mathbb{C}^{1 \times N}$$

ith element



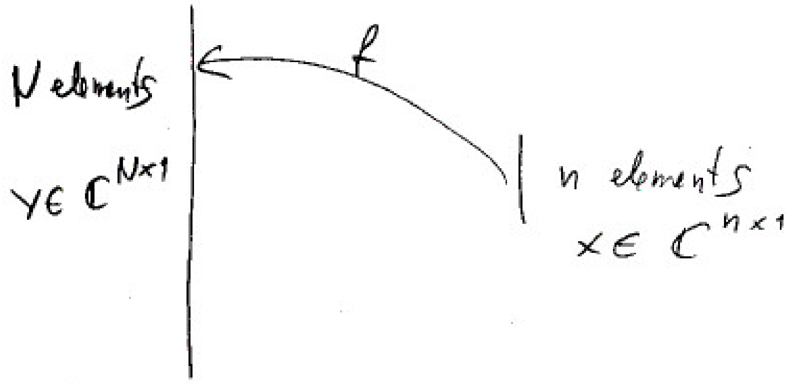
$$G_f = \begin{bmatrix} \left(e_{f(0)}^N\right)^T \\ \hline \left(e_{f(1)}^N\right)^T \\ \vdots \\ \hline \left(e_{f(n-1)}^N\right)^T \end{bmatrix}$$

$$y = G_f x$$

Code for $y = G_f x$

```
for (i=0; i<n; i++)
  y[i] = x[f(i)];
```

Scatter Matrices



$$S_f = \left[\, e^N_{f(0)} \;\middle|\; e^N_{f(1)} \;\middle|\; \dots \;\middle|\; e^N_{f(n-1)} \,\right]$$

Code for $y = S_f\, x$

```
for (i=0; i<n; i++)
    y[f(i)] = x[i];
```

Permutation Matrices

permutation function

$p : \mathbb{I}_n \to \mathbb{I}_n ; \; i \mapsto p(i),$ bijective

Shortcut: $p^{n\sigma}$

$\mathrm{perm}(p^{n\sigma}) := \sigma_p$

Code for $y = \mathrm{perm}(p^{n\sigma})x$

```
for (i=0; i<n; i++)
    y[i] = x[p(i)];
```

Example: L^{mn}_m

$L^{mn}_m = \mathrm{perm}(\ell^{mn}_m)$

$$\ell^{mn}_m : \mathbb{I}_{mn} \to \mathbb{I}_{mn} ; \; i \mapsto \begin{cases} (im) \bmod (mn-1) & , \; i < mn-1 \\ mn-1 & , \; i = mn-1 \end{cases}$$

equivalent to $\qquad i \mapsto \left\lfloor \frac{i}{n} \right\rfloor + m\,(i \bmod m)$

Example continued

$I_2 \otimes F_2$

$$G_0 = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) = G_{f_0} \qquad \rightarrow \quad f_0 = i \mapsto i \qquad 0 \le i < 2$$

$$f_1 = i \mapsto i+2 \qquad 0 \le i < 2$$

$$G_1 = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) = G_{f_1}$$

$$f_j = i \mapsto i + 2j \qquad \begin{array}{l} 0 \le i < 2 \\ 0 \le j < 2 \end{array}$$

$$S_0 = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right) = S_{f_0}$$

$$f_j : \mathbb{I}_2 \to \mathbb{I}_4;\ i \mapsto i + 2j$$

$$S_1 = \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right) = S_{f_1}$$

$$I_2 \otimes F_2 = \sum_{j=0}^{1} S_j F_2 G_j$$

$$= \sum_{j=0}^{1} S_{f_j} F_2 G_{f_j} = \sum_{j=0}^{1} S_{i \mapsto i+2j}\, F_2\, G_{i \mapsto i+2j}$$

$(I_2 \otimes F_2) L^4_2$

$$G_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = G_{g_0} \qquad g_0 : i \mapsto 2i \qquad 0 \le i < 2$$

$$G_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = G_{g_1} \qquad g_1 : i \mapsto 2i+1 \qquad 0 \le i < 2$$

$$g_j : i \mapsto 2i + j \qquad \begin{array}{l} 0 \le i < 2 \\ 0 \le j < 2 \end{array}$$

$$g_j : \mathbb{I}_2 \to \mathbb{I}_4;\ i \mapsto 2i + j$$

$$(I_2 \otimes F_2) L^4_2 = \sum_{j=0}^{1} S_{f_j} F_2 G_{g_j} = \sum_{j=0}^{1} S_{i \mapsto i+2j}\, F_2\, G_{i \mapsto 2i+j}$$

Code from Σ-SPL

$y = (I_2 \otimes F_2) L^4_2\, x$

↓ Formula manipulation

$$y = \sum_{j=0}^{1} S_{i \mapsto i+2j}\, F_2\, G_{i \mapsto 2i+j}$$

↓ Σ-SPL to Code: unroll F_2, i-loop in G, S; loop j-loop in Σ, G, S

```
for (j = 0; j < 2; j++) {
    t0 = x[j];
    t1 = x[j+2];
    u0 = t0 + t1;
    u1 = t0 - t1;
    y[2j] = u0;
    y[2j+1] = u1;
}
```

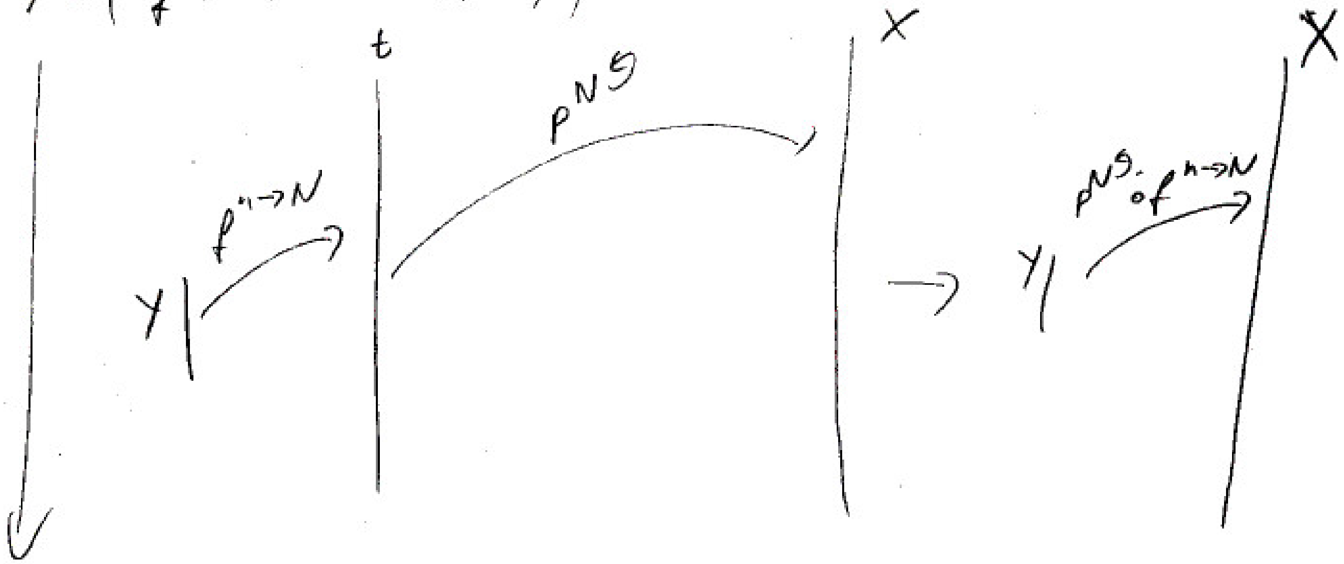
↓ Copy propagation

```
for (j = 0; j < 2; j++) {
    y[2j] = x[j] + x[j+2];
    y[2j+1] = x[j] - x[j+2];
}
```

Permutations and Gather/Scatter

Gather:

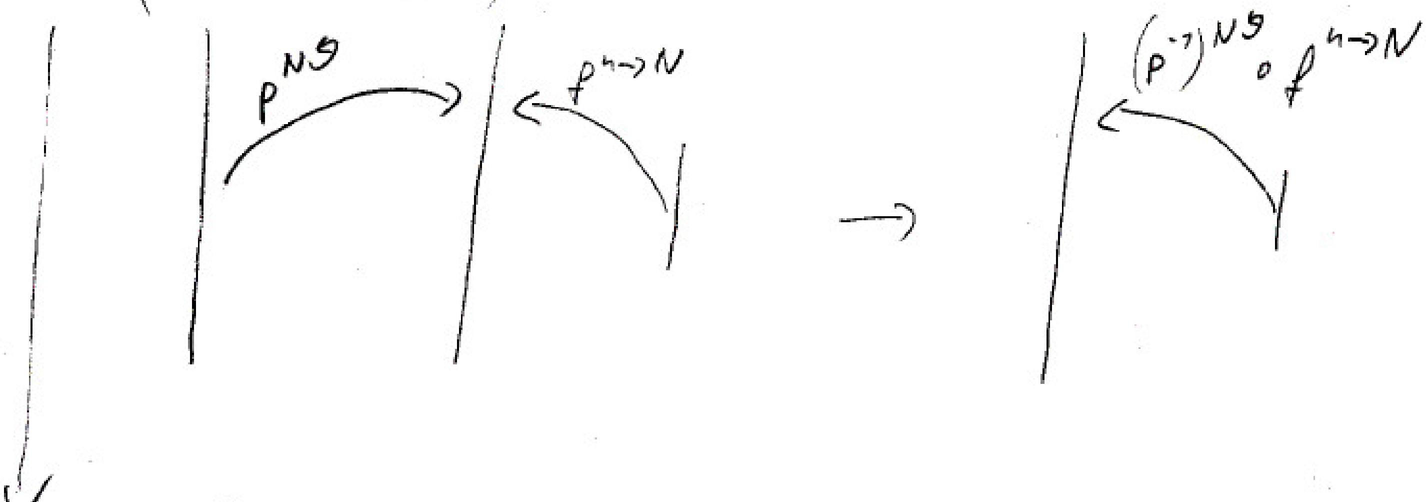
$$Y = \left(G_{f^{n\to N}} \cdot \mathrm{perm}\left(p^{NS}\right)\right) X$$



$$Y = \left(G_{p^{NS} \circ f^{n\to N}}\right) X$$

Scatter:

$$Y = \left(\mathrm{perm}\left(p^{NS}\right) S_{f^{n\to N}}\right) X$$



$$Y = S_{(p^{-1})^{NS} \circ f^{n\to N}} \cdot X$$

Handle Permutations: Example

Have: $(I_2 \otimes F_2) L^4_2$

want: $\sum_{j=0}^{1} S_{f_j} F_2 G_{g_j}$

$(I_2 \otimes F_2) L^4_2 =$

translate to Σ-SPL

$$\left(\sum_{j=0}^{1} S_{i\mapsto i+2j} F_2 G_{i\mapsto i+2j}\right) \mathrm{perm}\left(i \mapsto \begin{cases} 2i \bmod 3 & ; i<3 \\ 3 & ; i=3 \end{cases}\right) =$$

| distributivity |=

$$\sum_{j=0}^{1} S_{i\mapsto i+2j} F_2 \left(G_{i\mapsto i+2j} \cdot \mathrm{perm}(\ldots)\right)$$

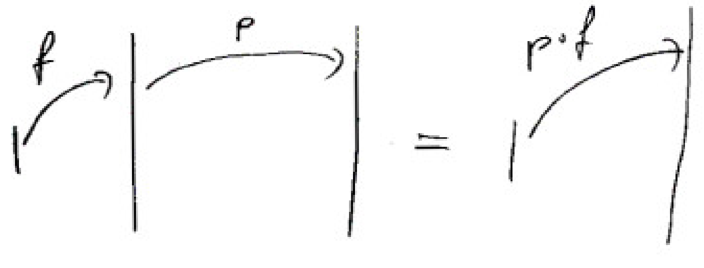
| Σ-SPL rule: $G_f \cdot \mathrm{perm}(p) = G_{p\circ f}$ |

$$= \sum_{j=0}^{1} S_{i\mapsto i+2j} F_2 \; G_{i\mapsto \begin{cases} (2(i+2j)) \bmod 3 & ; \text{else} \\ 3 & ; i=1, j=1 \end{cases}}$$

| algebraic simplification |

$$= \sum_{j=0}^{1} S_{i\mapsto i+2j} F_2 \; G_{i\mapsto 2i+j}$$

hard problem



Symbolic Functions

$\iota_n : \mathbb{I}_n \to \mathbb{I}_n;\ i \mapsto i$

$(j)_n : \mathbb{I}_1 \to \mathbb{I}_n;\ i \mapsto j$

Function Operators

$f : \mathbb{I}_m \to \mathbb{I}_M;\ i \mapsto f(i)$

$g : \mathbb{I}_n \to \mathbb{I}_N;\ i \mapsto g(i)$

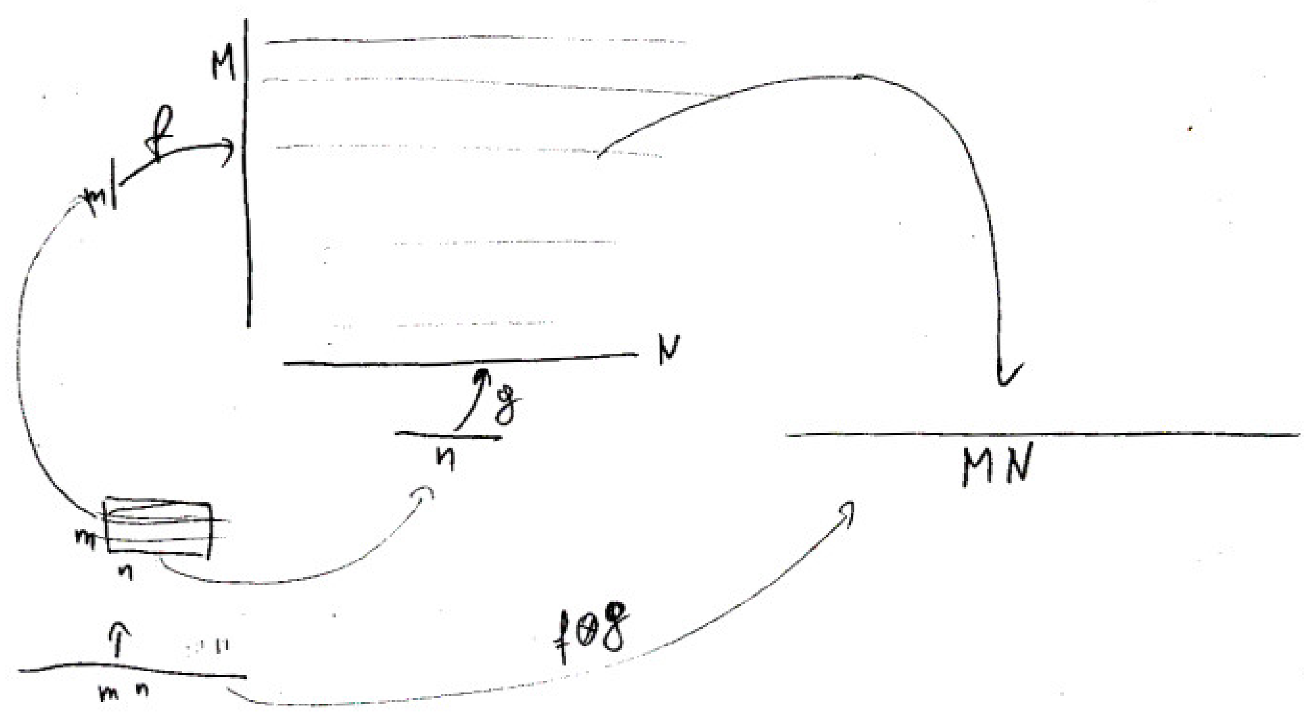
Associativity:

$(f \circ g) \circ h = f \circ (g \circ h)$

$(f \otimes g) \otimes h = (f \otimes g) \otimes h$

for $n = M$: $\quad g \circ f : \mathbb{I}_m \to \mathbb{I}_N;\ i \mapsto g(f(i))$

$f \otimes g : \mathbb{I}_{mn} \to \mathbb{I}_{MN};\ i \mapsto N f\left(\left\lfloor \frac{i}{n} \right\rfloor\right) + g(i \bmod n)$



Example Function Operators

$f = \iota_{n'} : \mathbb{I}_{n'} \to \mathbb{I}_{n'} ; i \mapsto i$ $\quad m = n'$, $M = n'$ $\quad \iota_{n'}(i) = i, \quad 0 \le i < n'$

$g = (j)_{m'} : \mathbb{I}_1 \to \mathbb{I}_{m'} ; i \mapsto j$ $\quad n = 1$, $N = m'$ $\quad (j)_{n'}(i) = j, \quad 0 \le i < 1$

$$(\iota_{n'} \otimes (j)_{m'})(i) = (f \otimes g)(i) = N f\left(\left\lfloor \frac{i}{n} \right\rfloor\right) + g(\underbrace{i \bmod n}_{i \bmod 1 = 0}) =$$

$$m' \left\lfloor \frac{i}{1} \right\rfloor + \underbrace{g(0)}_{j} = m'i + j$$

$$\iota_{n'} \otimes (j)_{m'} : \mathbb{I}_{n'} \to \mathbb{I}_{m'n'} ; \quad i \mapsto m'i + j$$

$f = (j)_{m'} : \mathbb{I}_1 \to \mathbb{I}_{m'} ; \ i \mapsto j$ $\quad m = 1$, $M = m'$ $\quad (j)_{n'}(i) = j, \ 0 \le i < 1$

$g = \iota_{n'} : \mathbb{I}_{n'} \to \mathbb{I}_{n'} ; \ i \mapsto i$ $\quad n = n'$, $N = n'$ $\quad \iota_{n'}(i) = i, \ 0 \le i < n$

$$((j)_{m'} \otimes \iota_{n'})(i) = (f \otimes g)(i) = \underbrace{N f(\underbrace{\left\lfloor \tfrac{i}{n} \right\rfloor}_{0})}_{j} + g(\underbrace{i \bmod n}_{i,\ 0 \le i < n}) =$$

$$n'j + i$$

$$(j)_{m'} \otimes \iota_{n'} : \mathbb{I}_{n'} \to \mathbb{I}_{m'n'} ; \quad i \mapsto n'j + i$$

$$\left(f^{1 \to m} \otimes g^{n \to N}\right)(i) = n f(0) + g(i)$$

$$\left(g^{n \to N} \otimes f^{1 \to m}\right)(i) = m g(i) + f(0)$$

Symbolic Functions in Σ-SPL

$$I_m \otimes A_n = \sum_{j=0}^{m-1} S_{(j)_m \otimes \iota_n} A_n G_{(j)_m \otimes \iota_n}$$

$$A_n \otimes I_m = \sum_{j=0}^{m-1} S_{\iota_n \otimes (j)_m} A_n G_{\iota_n \otimes (j)_m}$$

$$(I_m \otimes A_n) L^{mn}_m = \sum_{j=0}^{m-1} S_{(j)_m \otimes \iota_n} A_n G_{\iota_n \otimes (j)_m}$$

$L^{mn}_m = \text{perm}(\ell^{mn}_m)$ with

$$\ell^{mn}_m : \mathbb{I}_{mn} \to \mathbb{I}_{mn};\ i \longmapsto \begin{cases} (im) \bmod (mn-1), & i < mn-1 \\ mn-1, & i = mn-1 \end{cases}$$

alternative definition

$$\ell^{mn}_m : \mathbb{I}_{mn} \to \mathbb{I}_{mn};\ i \mapsto \left\lfloor \frac{i}{n} \right\rfloor + m\,(i \bmod n)$$

Function Tensor Identities

$$\ell_m^{mn} \circ (f^{1\to m} \otimes g^{k\to n}) = g^{k\to n} \otimes f^{1\to m}$$

Proof:

$$(f^{1\to m} \otimes g^{k\to n})(i) = N f(\underbrace{\lfloor \tfrac{i}{n} \rfloor}_{=0}) + g(\underbrace{i \bmod n}_{=i}) = n f(0) + g(i) =: j$$

$$\begin{aligned}
\ell_m^{mn}(j) &= \lfloor \tfrac{j}{n} \rfloor + m (j \bmod n) \\
&= \left\lfloor \frac{n f(0) + g(i)}{n} \right\rfloor + m\big((n f(0) + g(i)) \bmod n\big) \\
&= f(0) + \underbrace{\left\lfloor \frac{g(i)}{n} \right\rfloor}_{=0} + m\Big(\underbrace{(\underbrace{n f(0) \bmod n}_{=0} + \underbrace{g(i) \bmod n}_{=g(i)})}_{<n} \bmod n\Big) \\
&= f(0) + m\, g(i) \\
&= m g(\lfloor \tfrac{i}{1} \rfloor) + f(i \bmod 1) = (g^{k\to n} \otimes f^{1\to m})(i)
\end{aligned}$$

$$(f^{k\to m} \otimes g^{1\to n}) \circ h^{r\to k} = (f \circ h)^{r\to m} \otimes g^{1\to n}$$

Proof:

$$\begin{aligned}
((f \otimes g) \circ h)(i) &= (f \otimes g)(h(i)) \\
&= n f(\underbrace{\lfloor \tfrac{h(i)}{1} \rfloor}_{h(i)}) + g(\underbrace{h(i) \bmod 1}_{0}) \\
&= n f(h(i)) + g(0) = n (f \circ h)(i) + g(0) \\
&= n (f \circ h)(\lfloor \tfrac{i}{1} \rfloor) + g(i \bmod 1) \\
&= ((f \circ h) \otimes g)(i)
\end{aligned}$$

$$(g^{1\to n} \otimes f^{k\to m}) \circ h^{r\to k} = g^{1\to n} \otimes (f \circ h)^{r\to m}$$

Pull in Stride Permutation

$$(I_m \otimes A_n) L^{mn}_m =$$

$$\left(\sum_{j=0}^{m-1} S_{(j)_m \otimes \iota_n} A_n G_{(j)_m \otimes \iota_n}\right) \mathrm{perm}(\ell^{mn}_m) =$$

$$\sum_{j=0}^{m-1} S_{(j)_m \otimes \iota_n} A_n \left(G_{(j)_m \otimes \iota_n} \mathrm{perm}(\ell^{mn}_m)\right) =$$

$$\sum_{j=0}^{m-1} S_{(j)_m \otimes \iota_n} A_n G_{\ell^{mn}_m \circ ((j)_m \otimes \iota_n)} =$$

$$\sum_{j=0}^{m-1} S_{(j)_m \otimes \iota_n} A_n G_{\iota_n \otimes (j)_m}$$

Special Case:

$$(I_2 \otimes F_2) L^4_2 = \sum_{j=0}^{1} S_{i \mapsto i+2j} F_2 G_{i \mapsto 2i+j}.$$

Multiple Recursion Steps

$$\Big(I_k \otimes \underbrace{(I_m \otimes A_n) L^{mn}_m}_{B_{mn}}\Big) L^{kmn}_k =$$

$$\sum_{r=0}^{k-1} S_{(r)_k \otimes \iota_{mn}} \underbrace{\left(\sum_{j=0}^{m-1} S_{(j)_m \otimes \iota_n} A_n G_{\iota_n \otimes (j)_m}\right)}_{B_{mn}} G_{\iota_{mn} \otimes (r)_k} =$$

Zoom in (rewriting):

$$G_{\iota_n \otimes (j)_m} \cdot G_{\iota_{mn} \otimes (r)_k} =$$

$$G(\iota_{mn} \otimes (r)_k) \circ (\iota_n \otimes (j)_m) =$$

$$G_{\iota_n \otimes (j)_m \otimes (r)_k} \quad \hat{=} \quad i \mapsto kmi + kj + r, \quad 0 \le i < n$$

$$S_{(r)_k \otimes \iota_{mn}} \cdot S_{(j)_m \otimes \iota_n} =$$

$$S_{((r)_k \otimes \iota_{mn}) \circ ((j)_m \otimes \iota_n)} =$$

$$S_{(r)_k \otimes (j)_m \otimes \iota_n} \quad \hat{=} \quad i \mapsto rmn + jn + i, \quad 0 \le i < n$$

$$= \sum_{r=0}^{k-1} \sum_{j=0}^{m-1} S_{(r)_k \otimes (j)_m \otimes \iota_n} A_n G_{\iota_n \otimes (j)_m \otimes (r)_k}$$

Code Example

$$y = \left(I_2 \otimes (I_2 \otimes F_2) L^4_2\right) L^8_2 \cdot x$$

$$\downarrow$$

$$y = \sum_{r=0}^{1} \sum_{j=0}^{1} S_{(r)_2 \otimes (j)_2 \otimes \iota_2} F_2 \, G_{\iota_2 \otimes (j)_2 \otimes (r)_2}$$

$$\downarrow$$

$$y = \sum_{r=0}^{1} \sum_{j=0}^{1} S_{i \mapsto 4r+2j+i} F_2 \, G_{i \mapsto 4i+2j+r}$$

$$\downarrow$$

```
for (r=0; r<=1; r++)
    for (j=0; j<=1; j++) {
        y[4r+2j] = x[2j+r] + x[4+2j+r];
        y[4r+2j+1] = x[2j+r] - x[4+2j+r];
    }
```

Diagonal Matrices

$$f^{n \to \mathbb{C}}$$

$$f : \mathbb{I}_n \to \mathbb{C};\quad i \mapsto f(i)$$

$$t_n^{mn} : i \mapsto \omega_{mn}^{\lfloor \frac{i}{n} \rfloor + m (\bmod n)}$$

$$T_n^{mn} = \operatorname{diag}\left(t_n^{mn}\right)$$

$$G_{f^{n\to N}} \cdot \operatorname{diag}\left(g^{N \to \mathbb{C}}\right) = \operatorname{diag}\left(g^{N\to\mathbb{C}} \circ f^{n\to N}\right) G_{f^{n\to N}}$$

$$\operatorname{diag}\left(g^{N\to\mathbb{C}}\right) S_{f^{n\to N}} = S_{f^{n\to N}} \operatorname{diag}\left(g^{N\to\mathbb{C}} \circ f^{n\to N}\right)$$

Example

$$\left(\mathrm{DFT}_m \otimes I_n\right) T_n^{mn} =$$

$$\left(\sum_{j=0}^{n-1} S_{\iota_m \otimes (j)_n} \mathrm{DFT}_m\, G_{\iota_m \otimes (j)_n}\right) \operatorname{diag}\left(t_n^{mn}\right) =$$

$$\sum_{j=0}^{n-1} S_{\iota_m \otimes (j)_n} \mathrm{DFT}_m \left(G_{\iota_m \otimes (j)_n} \operatorname{diag}\left(t_n^{mn}\right)\right) =$$

$$\sum_{j=0}^{n-1} S_{\iota_m \otimes (j)_n} \mathrm{DFT}_m \operatorname{diag}\left(t_n^{mn} \circ \left(\iota_m \otimes (j)_n\right)\right) G_{\iota_m \otimes (j)_n}$$

Cooley-Tukey FFT

$$\mathrm{DFT}_{mn} = (\mathrm{DFT}_m \otimes I_n)\, T_n^{mn}\, (I_n \otimes \mathrm{DFT}_n)\, L_m^{mn}$$

$$= \sum_{j=0}^{n-1} S_{\iota_m \otimes (j)_n}\, \mathrm{DFT}_m\, \mathrm{diag}\left(t_n^{mn} \circ (\iota_m \otimes (j)_n)\right)\, G_{\iota_m \otimes (j)_n} \, .$$

FFTW No-twiddle

$$\sum_{j=0}^{m-1} S_{(j)_m \otimes \iota_n}\, \mathrm{DFT}_n\, G_{\iota_n \otimes (j)_m}$$

FFTW recursion or no-Twiddle