Extending Spiral to Other Numerical Algorithms

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1 Extending Spiral's Formal Framework

1.1 Extending Σ-SPL to Nonlinear Operators

We extend [1] to general operators on complex vectors. We replace matrices by operators and matrix multiplication "·" by operator composition "∘".

Definition 1 (Operator)**.**

$$\mathrm{M} : \begin{cases} \mathbb{C}^{n_0} \times \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_{k-1}} \to \mathbb{C}^{N_0} \times \mathbb{C}^{N_1} \times \cdots \times \mathbb{C}^{N_{m-1}} \\ x \mapsto \mathrm{M}(x) \end{cases}$$

As shorthand we write

$$\mathrm{M}^{D \to R}.$$

"×" is associative:

$$\mathbb{C}^k \times \mathbb{C}^m \times \mathbb{C}^n \cong \big(\mathbb{C}^k \times \mathbb{C}^m\big) \times \mathbb{C}^n \cong \mathbb{C}^k \times \big(\mathbb{C}^m \times \mathbb{C}^n\big).$$

Notation 1.1 (Inplace Operator)**.** We denote an operator

$$\mathrm{M} : \begin{cases} \mathbb{C}^{n_0} \times \cdots \times \mathbb{C}^{n_{k-1}} \to \mathbb{C}^{n_0} \times \cdots \times \mathbb{C}^{n_{k-1}} \times \mathbb{C}^{N_k} \times \cdots \times \mathbb{C}^{N_{m-1}} \\ x \mapsto \mathrm{M}(x) \end{cases}$$

being inplace in its first k arguments by

$$\underbrace{\mathrm{M}}_{\text{inplace } (A_0,\dots,A_{m-1}) \to (A_0,\dots,A_{k-1})}$$

Definition 2 (Operation on Vectors)**.** A scalar operation

$$\diamond : \begin{cases} D_0 \times D_1 \times \cdots \times D_{n-1} \to D \\ (x_0, x_1, \dots, x_{n-1}) \mapsto \diamond(x_0, x_1, \dots, x_{n-1}) \end{cases}.$$

induces a vector operation

$$\diamond : \begin{cases} D_0^m \times \cdots \times D_{n-1}^m \to D^m \\ (x_0, \dots, x_{n-1}) \mapsto \big(\diamond(x_0^0, \dots, x_{n-1}^0), \dots, \diamond(x_0^{m-1}, \dots, x_{n-1}^{m-1})\big) \end{cases}.$$

Definition 3 (Operation on Operators)**.** For

$$\mathrm{M}_i : \begin{cases} D \to R_i \\ x \mapsto \mathrm{M}_i(x) \end{cases} \quad \text{and} \quad \diamond : \begin{cases} R_0 \times R_1 \times \cdots \times R_{n-1} \to R \\ (x_0, x_1, \dots, x_{n-1}) \mapsto \diamond(x_0, x_1, \dots, x_{n-1}) \end{cases}.$$

we define

$$\diamond(\mathrm{M}_0, \mathrm{M}_1 \dots, \mathrm{M}_{n\text{-}1}) : \begin{cases} D \to R \\ x \mapsto \diamond\big(\mathrm{M}_0(x), \mathrm{M}_1(x), \dots, \mathrm{M}_{n-1}(x)\big) \end{cases}.$$

We also write operations $\diamond$ in infix notation, e.g.,

$$\mathrm{M} \diamond \mathrm{N} : \begin{cases} D \to R \\ x \mapsto \mathrm{M}(x) \diamond \mathrm{N}(x) \end{cases} .$$

Definition 4 (Addition of Operators)**.** For

$$\mathrm{M} : \begin{cases} D \to R \\ x \mapsto \mathrm{M}(x) \end{cases} \quad , \quad \mathrm{N} : \begin{cases} D \to R \\ x \mapsto \mathrm{N}(x) \end{cases} \quad \text{we define} \quad \mathrm{M} + \mathrm{N} : \begin{cases} D \to R \\ x \mapsto \mathrm{M}(x) + \mathrm{N}(x) \end{cases} .$$

We also define the iterative sum

$$\sum_{i=0}^{n-1} M_i = M_0 + M_1 + \cdots + M_{n-1}$$

accordingly.

Definition 5 (Composition of Operators)**.** For

$$\mathrm{M} : \begin{cases} D \to S \\ x \mapsto \mathrm{M}(x) \end{cases} \quad , \quad \mathrm{N} : \begin{cases} S \to R \\ x \mapsto \mathrm{N}(x) \end{cases} \quad \text{we define} \quad \mathrm{N} \circ \mathrm{M} : \begin{cases} D \to R \\ x \mapsto \mathrm{N}(\mathrm{M}(x)) \end{cases} .$$

We denote the iterative composition of operators by

$$\prod_{i=0}^{n-1} M_i = M_0 \circ M_1 \circ \ldots \circ M_{n-1}$$

Definition 6 (Direct Sum of Operators)**.** For

$$\mathrm{M} : \begin{cases} D \to R \\ x \mapsto \mathrm{M}(x) \end{cases} \quad , \quad \mathrm{N} : \begin{cases} E \to S \\ x \mapsto \mathrm{N}(x) \end{cases} \quad \text{we define} \quad \mathrm{M} \oplus \mathrm{N} : \begin{cases} D \times E \to R \times S \\ (x, y) \mapsto (\mathrm{M}(x), \mathrm{N}(y)) \end{cases} .$$

We also define the iterative direct sum

$$\bigoplus_{i=0}^{n-1} M_i = M_0 \oplus M_1 \oplus \ldots \oplus M_{n-1}$$

accordingly.

Definition 7 (Parallel Evaluation of Operators)**.** For

$$\mathrm{M}_i : \begin{cases} D \to S_i \\ x \mapsto \mathrm{M}_i(x) \end{cases} \quad , \quad 0 \leq i < k$$

we define

$$\big(\mathrm{M}_0, \ldots, \mathrm{M}_{k\text{-}1}\big) : \begin{cases} D \to S_0 \times \cdots \times S_{k-1} \\ x \mapsto \big(\mathrm{M}_0(x), \ldots, \mathrm{M}_{k-1}(x)\big) \end{cases} .$$

Definition 8 (Projection Operation)**.**

$$\pi^{(A_0, \ldots, A_{k-1}) \to (A_{n_0}, \ldots, A_{n_{m-1}})} : \begin{cases} \mathbb{C}^{N_0} \times \cdots \times \mathbb{C}^{N_{k-1}} \to \mathbb{C}^{N_{n_0}} \times \cdots \times \mathbb{C}^{N_{n_{m-1}}} \\ (x_0, \ldots, x_{k-1}) \mapsto (x_{n_0}, \ldots, x_{n_{m-1}}) \end{cases} .$$

As shorthand notation we write for instance

$$(A, B, C) \to A$$

Definition 9 (Gather Operator)**.** The gather operator is defined via matrix-vector product with a gather matrix,

$$\mathrm{G}_{f^{n\to N}} : \begin{cases} \mathbb{C}^N \to \mathbb{C}^n \\ x \mapsto \mathrm{G}_f\, x \end{cases} .$$

Definition 10 (Scatter Operator)**.** The scatter operator is defined via matrix-vector product with a scatter matrix,

$$\mathrm{S}_{f^{n\to N}} : \begin{cases} \mathbb{C}^n \to \mathbb{C}^N \\ x \mapsto \mathrm{S}_f\, x \end{cases} .$$

1.2 Rewrite Rules

Rule 1.1 (Pull into Operation)**.**

$$\diamond\big(A_0, \dots, A_{k-1}\big) \circ B \to \diamond\big(A_0 \circ B, \dots, A_{k-1} \circ B\big)$$

Rule 1.2 (Drop Projection)**.**

$$\pi^{(a_0,\dots,a_{k-1})\to a_i} \circ \big(A_0, \dots, A_{k-1}\big) \to A_i$$

Rule 1.3 (Pull into Iterative Composition)**.**

$$\left(\prod_{i=0}^{N-1} \big(A_{i,0}, \dots, A_{i,k-1}\big)\right) \circ \big(B_0, \dots, B_{k-1}\big) \to \prod_{i=0}^{N-1} \big(A_{i,0} \circ B_0, \dots, A_{i,k-1} \circ B_{k-1}\big)$$

for $r \in \{0, \dots, k-1\}$ and

$$A_{i,j} = \begin{cases} C_{i,j}^{n_0,\dots,n_{k-1}\to n_j}, & j = r \\ \pi^{(a_0,\dots,a_{k-1})\to a_j} & j \neq r \end{cases} \quad \text{and} \quad B_j = \begin{cases} \pi^{(a_0,\dots,a_{k-1})\to a_j} & j = r \\ D_j \circ \pi^{(a_0,\dots,a_{k-1})\to a_j} & j \neq r \end{cases} .$$

Rule 1.4 (Flatten Iterative Composition)**.**

$$\big(A_0, \dots, A_{k-1}\big) \to \prod_{i=0}^{N-1} \big(B_{i,0}, \dots, B_{i,k-1}\big)$$

for $r \in \{0, \dots, k-1\}$ and

$$A_j = \begin{cases} \pi^{(a_0,\dots,a_{k-1})\to a_j} \circ \prod_{i=0}^{N-1} \big(B_{i,0}, \dots, B_{i,k-1}\big), & j = r \\ \pi^{(a_0,\dots,a_{k-1})\to a_j} & j \neq r \end{cases} \quad \text{with} \quad B_{u,v} = \begin{cases} C_{u,v}^{n_0,\dots,n_{k-1}\to n_v}, & v = r \\ \pi^{(a_0,\dots,a_{k-1})\to a_v} & v \neq r \end{cases} .$$

1.3 Nonterminals and Breakdown Rules

Definition 11 (Nonterminal)**.** A parameterized operator

$$\mathrm{M}(p) : \begin{cases} D(p) \to R(p) \\ x \mapsto \big(\mathrm{M}(p)\big)(x) \end{cases} , \quad p \in P$$

is a nonterminal with a parameter space P.

Definition 12 (Breakdown Rule)**.** Let $\mathrm{M}(p), p \in P$ and $\mathrm{N}_i(p_i), p_i \in P_i$,

$$\mathrm{M}(p) : \begin{cases} D(p) \to R(p) \\ x \mapsto \big(\mathrm{M}(p)\big)(x) \end{cases} , \quad \mathrm{N}_i(p_i) : \begin{cases} D_i(p_i) \to R_i(p_i) \\ x \mapsto \big(\mathrm{N}_i(p_i)\big)(x) \end{cases} , \quad 0 \leq i < k$$

be nonterminals. Let $\mathcal{F}_{n_0,\dots,n_{m-1}}(M_0,\dots,M_{k-1})$ be an expression of nonterminals, operators, and operations. We define the set of all valid breakdowns for a parameter p

$$\mathrm{X}_p = \{\chi, \xi, \dots\}$$

with

$$\chi, \xi, \dots \in P_0 \times \dots \times P_{k-1}.$$

A breakdown rule is a mapping

$$\mathrm{rule}_{p,\chi} : \begin{cases} M(p) \to \big(D(p) \to R(p)\big) \\ \mathrm{M} \mapsto \mathcal{F}_{f_0(p),\dots,f_{m-1}(p)}\big(N_0(\chi_0),\dots,N_{k-1}(\chi_{k-1})\big) \end{cases} \quad , \quad p \in P,\ \chi \in \mathrm{X}_p$$

with

$$\big(\mathrm{M}(p)\big)(x) = \Big(\mathcal{F}_{f_0(p),\dots,f_{m-1}(p)}\big(N_0(\chi_0),\dots,N_{k-1}(\chi_{k-1})\big)\Big)(x) \quad \forall x \in D(p).$$

2 Linear DSP Transforms

2.1 Cooley-Tukey FFT

We define

$$\mathrm{M}(p) = \mathrm{DFT}_p : \begin{cases} \mathbb{C}^p \to \mathbb{C}^p \\ x \mapsto \mathrm{DFT}_p\, x \end{cases}$$

and $P = \{km\}$ with $k, m \in \mathbb{N}$, and $\mathrm{N}_i(p_i) = \mathrm{M}(p_i), 0 = 1, 2$. We define the set of all valid Cooley-Tukey FFT breakdowns

$$\mathrm{X}_n = \big\{(k,m) \mid n = km\big\}.$$

Rule 2.1 (Cooley-Tukey FFT)**.** The Cooley-Tukey rule parameterized by a breakdown χ is given by

$$\mathrm{CT}_\chi^n : \begin{cases} \mathrm{DFT}_n \to \big(\mathbb{C}^n \to \mathbb{C}^n\big) \\ \mathrm{M} \mapsto \mathcal{F}_{\chi_0,\chi_1}\big(\mathrm{N}_0(\chi_0), \mathrm{N}_1(\chi_1)\big) \end{cases} \qquad \text{with} \quad n \in P,\ \chi \in \mathrm{X}_n$$

with

$$\mathcal{F}_{\chi_0(n),\chi_1(n)}\big(\dots\big) = \left(\sum_{i=0}^{m-1} \mathrm{S}_{\imath_k \otimes (i)_m} \circ \mathrm{DFT}_k \circ \mathrm{G}_{\imath_k \otimes (i)_m}\right) \circ \mathrm{T}_m^{km} \circ \left(\sum_{i=0}^{k-1} \mathrm{S}_{(i)_k \otimes \imath_m} \circ \mathrm{DFT}_m \circ \mathrm{G}_{\imath_m \otimes (i)_k}\right)$$

and

$$\mathrm{T}_m^{km} : \begin{cases} \mathbb{C}^{km} \to \mathbb{C}^{km} \\ x \mapsto \mathrm{T}_m^{km}\, x \end{cases} .$$

Rule 2.2 (DFT Base Rule)**.** The base rule is

$$\mathrm{base} : \begin{cases} \mathrm{DFT}_2 \to \big(\mathbb{C}^2 \to \mathbb{C}^2\big) \\ \mathrm{M} \mapsto \big(x \mapsto \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right] x\big) \end{cases}$$

with $\mathrm{X}_n = \emptyset$ and no $\mathrm{N}_i(p)$ defined.

2.2 Convolution

We define

$$\mathrm{M}(p) = \mathrm{Conv}_p : \begin{cases} \mathbb{C}^p \times \mathbb{C}^p \to \mathbb{C}^p \\ (x,y) \mapsto \mathrm{Conv}_p(x,y) \end{cases}$$

and

$$P = \mathbb{N} \quad , \quad \mathrm{N}_0(p) = \mathrm{DFT}_p \quad , \quad \mathrm{N}_1(p) = \mathrm{DFT}_p \quad , \quad \mathrm{N}_2(p) = \mathrm{DFT}_p^{-1} \quad , \quad \mathrm{X}_n = \{n\}.$$

Rule 2.3 (Convolution via DFT)**.** The convolution breakdown rule is given by

$$\mathrm{conv}_\chi^n : \begin{cases} \mathrm{Conv}_n \to \big(\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n\big) \\ \mathrm{M} \mapsto \mathcal{F}_n\big(\,\mathrm{N}_0(\chi), \mathrm{N}_1(\chi), \mathrm{N}_2(\chi)\big) \end{cases} \qquad \text{with} \quad n \in P,\ \chi \in \mathrm{X}_n$$

with

$$\mathcal{F}_n\big(\,\mathrm{N}_0(\chi), \mathrm{N}_1(\chi), \mathrm{N}_2(\chi)\big) = \mathrm{DFT}_n^{-1} \circ \Big(\big(\,\mathrm{DFT}_n \circ \mathrm{G}_{\imath_n}^{(x,y)\to y}\,\big) \cdot \big(\,\mathrm{DFT}_n \circ \mathrm{G}_{\imath_n}^{(x,y)\to x}\,\big)\Big).$$

3 ATLAS Matrix-Matrix-Multiply

Throughout this section we follow [2].

3.1 Nonterminals and Breakdown Rules

We use

$$\pi_A = \pi^{(C,A,B)\to A}$$
$$\pi_B = \pi^{(C,A,B)\to B}$$
$$\pi_C = \pi^{(C,A,B)\to C}$$

throughout this section.

Notation 3.1 (Reshaping)**.** Reshaping can be used to define nonterminals for matrix operations.

$$[.]^{\mathrm{m}_0\times \mathrm{m}_1\times \ldots \mathrm{m}_{r\text{-}1}\to \mathrm{n}_0\times \mathrm{n}_1\times \ldots \mathrm{n}_{s\text{-}1}} : \begin{cases} \mathbb{C}^{m_0\times m_1\times \ldots m_{r-1}} \to \mathbb{C}^{n_0\times n_1\times \ldots n_{r-1}} \\ a \mapsto [a]^{m_0\times m_1\times \ldots m_{r-1}\to n_0\times n_1\times \ldots n_{s-1}} \end{cases} \qquad \text{with} \quad \prod_{i=0}^{r-1} m_i = \prod_{i=0}^{s-1} n_i$$

For instance, to interpret a data vector $a \in \mathbb{C}^{mn}$ as a matrix $A \in \mathbb{C}^{m\times n}$ stored in row-major order we write

$$A = [a]^{mn\to m\times n}.$$

Definition 13 (Matrix-Matrix-Multiply Nonterminal)**.** We define the matrix-matrix-multiply nonterminal $C+ = AB$ by

$$\mathrm{MMM}_{f^{N\to N'}\otimes g^{M\to M'}}^{N,M,K} : \begin{cases} \mathbb{C}^{N'M'} \times \mathbb{C}^{NK} \times \mathbb{C}^{KM} \to \mathbb{C}^{N'M'} \\ (C,A,B) \mapsto C + \mathrm{S}_{f\otimes g}\Big[[A]^{NK\to N\times K}\,[B]^{KM\to K\times M}\Big]^{N\times M\to NM} \end{cases}.$$

$p = (N, M, K)$, $N \leq N'$, $M \leq M'$, and the parameter space $P_{f,g} \subset \mathbb{N}^3$.

Rule 3.1 (Base Rule)**.** The MMM base rule is

$$\mathrm{base} : \begin{cases} \mathrm{MMM}_{f^{1\to N'}\otimes g^{1\to M'}}^{1,1,1} \to \big(\mathbb{C}^{N'M'} \times \mathbb{C}^1 \times \mathbb{C}^1 \to \mathbb{C}^{N'M'}\big) \\ \mathrm{M} \mapsto \pi_C + \big(\mathrm{S}_{f\otimes g} \circ (\pi_A \cdot \pi_B)\big) \end{cases}.$$

Rule 3.2 (Tiling Rule)**.** We define

$$\mathrm{M}(N,M,K) = \mathrm{MMM}_{f^{N\to N'}\otimes g^{M\to M'}}^{N,M,K} \quad \text{and} \quad \mathrm{N}(N_B,M_B,K_B) = \mathrm{MMM}_{(f\otimes g)\circ\big((i)_{N/N_B}\otimes \imath_{N_B}\otimes (j)_{M/M_B}\otimes \imath_{M_B}\big)}^{N_B,M_B,K_B}$$

and

$$P_{f^{N\to N'},g^{M\to M'}} = \big\{n \in \mathbb{N}^3 \mid n_0 \leq N', n_1 \leq M'\big\}.$$

We define the set of all valid tilings (blockings)

$$\mathrm{X}_{N,M,K} = \big\{(N_B, M_B, K_B) \mid N = nN_B, M = mM_B, K = kK_B\big\}.$$

All loop orders are given by

$$\Gamma = \Big\{\big(\pi(i), \pi(j), \pi(k)\big) \mid \pi \in S_3\Big\}$$

and we define the ranges of i, j, and k,

$$\rho : \begin{cases} \{i,j,k\} \to \{N/N_B, M/M_B, K/K_B\} \\ x \mapsto \begin{cases} N/N_B & \text{if } x = i \\ M/M_B & \text{if } x = j \\ K/K_B & \text{if } x = k \end{cases} \end{cases}.$$

Then tiling is given by

$$\text{tile}_{\chi,\gamma}^{N,M,K} : \begin{cases} \text{MMM}_{f^{N\to N'}\otimes g^{M\to M'}}^{N,M,K} \to \big(\mathbb{C}^{N'M'} \times \mathbb{C}^{NK} \times \mathbb{C}^{KM} \to \mathbb{C}^{N'M'}\big) \\ \text{M} \mapsto \mathcal{F}_{N,M,K,\chi,\gamma}\big(\text{N}(\chi)\big) \end{cases} \quad \text{with} \quad \begin{matrix} (N,M,K) \in P_{f,g} \\ \chi \in \text{X}_{N,M,K} \\ \gamma \in \Gamma \end{matrix}$$

and

$$\mathcal{F}_{N,M,K,\chi,\gamma}\big(\text{N}(\chi)\big) = \pi_C \circ \left(\prod_{\gamma_0=0}^{\rho(\gamma_0)-1} \prod_{\gamma_1=0}^{\rho(\gamma_1)-1} \prod_{\gamma_2=0}^{\rho(\gamma_2)-1} \big(M_{i,j,k} \circ G_{i,j,k},\ \pi_A,\ \pi_B\big) \right)$$

and

$$\begin{aligned} M_{i,j,k} &= \text{MMM}_{(f\otimes g)\circ\left((i)_{N/N_B}\otimes\imath_{N_B}\otimes(j)_{M/M_B}\otimes\imath_{M_B}\right)}^{N_B,M_B,K_B} \\ G_{i,j,k} &= \Big(\pi_C,\ \text{G}_{(i)_{N/N_B}\otimes\imath_{N_B}\otimes(k)_{K/K_B}\otimes\imath_{K_B}} \circ\pi_A,\ \text{G}_{(k)_{K/K_B}\otimes\imath_{K_B}\otimes(j)_{M/M_B}\otimes\imath_{M_B}} \circ\pi_B\Big). \end{aligned}$$

3.2 Expressing ATLAS in Σ-SPL

Example 3.1 (Naive MMM Implementation)**.** We now obtain the naive implementation of

$$C^{N\times M} += A^{N\times K} B^{K\times M}$$

using rule (3.1) and (3.2). We start with the nonterminal

$$\underbrace{\text{MMM}_{\imath_N\otimes\imath_M}^{N,M,K}}_{\text{inplace }(C,A,B)\to C}$$

and break down using rule (3.2) with

$$\chi = (1,1,1) \quad \text{and} \quad \gamma = (i,j,k)$$

leading to

$$\underbrace{\text{MMM}_{\imath_N\otimes\imath_M}^{N,M,K}}_{\text{inplace }(C,A,B)\to C} \to \underbrace{\pi_C \circ \left(\prod_{i=0}^{N-1} \prod_{j=0}^{M-1} \prod_{k=0}^{K-1} \Big(\text{MMM}_{(i)_N\otimes(j)_M}^{1,1,1} \circ \Big(\pi_C,\ \text{G}_{(i)_N\otimes(k)_K} \circ\pi_A,\ \text{G}_{(k)_K\otimes(j)_M} \circ\pi_B\Big),\ \pi_A,\ \pi_B\Big)\right)}_{\text{inplace }(C,A,B)\to C}.$$

Now we further break down using rule (3.1) expanding

$$\text{MMM}_{(i)_N\otimes(j)_M}^{1,1,1} \to \pi_C + \text{S}_{(i)_N\otimes(j)_M} \circ\big(\pi_A\cdot\pi_B\big)$$

leading to

$$\underbrace{\text{MMM}_{\imath_N\otimes\imath_M}^{N,M,K}}_{\text{inplace }(C,A,B)\to C} \to \underbrace{\pi_C \circ \left(\prod_{i=0}^{N-1} \prod_{j=0}^{M-1} \prod_{k=0}^{K-1} C_{i,j,k} \right)}_{\text{inplace }(C,A,B)\to C}$$

with

$$C_{i,j,k} = \Big(\big(\pi_C + \text{S}_{(i)_N\otimes(j)_M} \circ\big(\pi_A\cdot\pi_B\big)\big) \circ \big(\pi_C,\ \text{G}_{(i)_N\otimes(k)_K} \circ\pi_A,\ \text{G}_{(k)_K\otimes(j)_M} \circ\pi_B\big),\ \pi_A,\ \pi_B\Big).$$

Using rewrite rule (1.1) we now rewrite $C_{i,j,k}$ into

$$\begin{aligned} C_{i,j,k} &= \Big(\pi_C \circ \big(\pi_C,\, \mathrm{G}_{(i)_N \otimes (k)_K} \circ \pi_A,\, \mathrm{G}_{(k)_K \otimes (j)_M} \circ \pi_B\big) + \\ &\quad + \mathrm{S}_{(i)_N \otimes (j)_M} \circ \big(\pi_A \cdot \pi_B\big) \circ \big(\pi_C,\, \mathrm{G}_{(i)_N \otimes (k)_K} \circ \pi_A,\, \mathrm{G}_{(k)_K \otimes (j)_M} \circ \pi_B\big),\, \pi_A,\, \pi_B\Big) \end{aligned}$$

and further

$$\begin{aligned} C_{i,j,k} &= \Big(\pi_C \circ \big(\pi_C,\, \mathrm{G}_{(i)_N \otimes (k)_K} \circ \pi_A,\, \mathrm{G}_{(k)_K \otimes (j)_M} \circ \pi_B\big) + \\ &\quad + \mathrm{S}_{(i)_N \otimes (j)_M} \circ \Big(\big(\pi_A \circ \big(\pi_C,\, \mathrm{G}_{(i)_N \otimes (k)_K} \circ \pi_A,\, \mathrm{G}_{(k)_K \otimes (j)_M} \circ \pi_B\big)\big) \cdot \\ &\quad \cdot \big(\pi_B \circ \big(\pi_C,\, \mathrm{G}_{(i)_N \otimes (k)_K} \circ \pi_A,\, \mathrm{G}_{(k)_K \otimes (j)_M} \circ \pi_B\big)\big)\Big), \\ &\quad \pi_A,\, \pi_B\Big). \end{aligned}$$

Applying rule (1.2) leads to

$$C_{i,j,k} = \Big(\pi_C + \mathrm{S}_{(i)_N \otimes (j)_M} \circ \big((\mathrm{G}_{(i)_N \otimes (k)_K} \circ \pi_A) \cdot (\mathrm{G}_{(k)_K \otimes (j)_M} \circ \pi_B)\big),\, \pi_A,\, \pi_B\Big).$$

The final expression

$$\underbrace{\mathrm{MMM}^{N,M,K}_{\imath_N \otimes \imath_M}}_{\text{inplace } (C,A,B)\to C} \rightarrow \underbrace{\pi_C \circ \left(\prod_{i=0}^{N-1} \prod_{j=0}^{M-1} \prod_{k=0}^{K-1} \Big(\pi_C + \mathrm{S}_{(i)_N \otimes (j)_M} \circ \big((\mathrm{G}_{(i)_N \otimes (k)_K} \circ \pi_A) \cdot (\mathrm{G}_{(k)_K \otimes (j)_M} \circ \pi_B)\big),\, \pi_A,\, \pi_B\Big)\right)}_{\text{inplace } (C,A,B)\to C}$$

is the Σ-SPL equivalent to

$$\begin{aligned} &\texttt{for } i \in [0:1:N-1] \\ &\quad \texttt{for } j \in [0:1:M-1] \\ &\quad\quad \texttt{for } k \in [0:1:K-1] \\ &\quad\quad\quad C_{i,j} = C_{i,j} + A_{ik}B_{kj}. \end{aligned}$$

Example 3.2 (Tiling in ATLAS)**.** We now obtain the two-level ATLAS tiling (L1 and register tiling) implementation of

$$C^{N\times M} += A^{N\times K} B^{K\times M}$$

using rule (3.1) and (3.2).
L1 tiling. We start with the nonterminal

$$\underbrace{\mathrm{MMM}^{N,M,K}_{\imath_N \otimes \imath_M}}_{\text{inplace } (C,A,B)\to C} \quad , \quad N_B \mid N, K, M;\ M_U \mid N/N_B;\ N_U \mid M/N_B;\ K_U \mid K/N_B$$

and break down using rule (3.2) with

$$\chi = (N_B, N_B, N_B) \quad \text{and} \quad \gamma = (j, i, k)$$

leading to

$$\underbrace{\mathrm{MMM}^{N,M,K}_{\imath_N \otimes \imath_M}}_{\text{inplace } (C,A,B)\to C} \rightarrow \underbrace{\pi_C \circ \left(\prod_{j=0}^{N/N_B-1} \prod_{i=0}^{M/N_B-1} \prod_{k=0}^{K/N_B-1} (M_{i,j,k} \circ G_{i,j,k},\, \pi_A,\, \pi_B)\right)}_{\text{inplace } (C,A,B)\to C}$$

with

$$\begin{aligned} M_{i,j,k} &= \mathrm{MMM}^{N_B,N_B,N_B}_{(i)_{N/N_B} \otimes \imath_{N_B} \otimes (j)_{M/N_B} \otimes \imath_{N_B}} \\ G_{i,j,k} &= \Big(\pi_C,\, \mathrm{G}_{(i)_{N/N_B} \otimes \imath_{N_B} \otimes (k)_{K/N_B} \otimes \imath_{N_B}} \circ \pi_A,\, \mathrm{G}_{(k)_{K/N_B} \otimes \imath_{N_B} \otimes (j)_{M/N_B} \otimes \imath_{N_B}} \circ \pi_B\Big). \end{aligned}$$

Register blocking. Next we break down

$$\mathrm{MMM}^{N_B,N_B,N_B}_{(i)_{N/N_B}\otimes\imath_{N_B}\otimes(j)_{M/N_B}\otimes\imath_{N_B}}$$

using rule (3.2) with

$$\chi = (M_U, N_U, K_U) \quad \text{and} \quad \gamma = (j, i, k)$$

leading to

$$M_{i,j,k} \to \pi_C \circ \left(\prod_{j'=0}^{N_B/N_U-1} \prod_{i'=0}^{N_B/M_U-1} \prod_{k'=0}^{N_B/K_U-1} \left(M'_{i,j,k,i',j',k'} \circ G'_{i',j',k'},\, \pi_A,\, \pi_B \right) \right)$$

with

$$\begin{aligned} M'_{i,j,k,i',j',k'} &= \mathrm{MMM}^{M_U,N_U,K_U}_{(i)_{N/N_B}\otimes(i')_{N_B/M_U}\otimes\imath_{M_U}\otimes(j)_{M/N_B}\otimes(j')_{N_B/N_U}\otimes\imath_{N_U}} \\ G'_{i',j',k'} &= \Big(\pi_C,\, \mathrm{G}_{(i')_{N_B/M_U}\otimes\imath_{M_U}\otimes(k')_{N_B/K_U}\otimes\imath_{K_U}} \circ\pi_A,\, \mathrm{G}_{(k')_{N_B/K_U}\otimes\imath_{K_U}\otimes(j')_{N_B/N_U}\otimes\imath_{N_U}} \circ\pi_B\Big). \end{aligned}$$

Unrolling. Next we break down

$$\mathrm{MMM}^{M_U,N_U,K_U}_{(i)_{N/N_B}\otimes(i')_{N_B/M_U}\otimes\imath_{M_U}\otimes(j)_{M/N_B}\otimes(j')_{N_B/N_U}\otimes\imath_{N_U}}$$

using rule (3.2) with

$$\chi = (1, 1, 1) \quad \text{and} \quad \gamma = (k, j, i)$$

leading to

$$M'_{i,j,k,i',j',k'} \to \pi_C \circ \left(\prod_{k''=0}^{K_U-1} \prod_{j''=0}^{N_U-1} \prod_{i''=0}^{M_U-1} \left(M''_{i,j,k,i',j',k',i'',j'',k''} \circ G''_{i'',j'',k''},\, \pi_A,\, \pi_B \right) \right)$$

with

$$\begin{aligned} M''_{i,j,k,i',j',k',i'',j'',k''} &= \mathrm{MMM}^{1,1,1}_{(i)_{N/N_B}\otimes(i')_{N_B/M_U}\otimes(i'')_{M_U}\otimes(j)_{M/N_B}\otimes(j')_{N_B/N_U}\otimes(j'')_{N_U}} \\ G''_{i'',j'',k''} &= \Big(\pi_C,\, \mathrm{G}_{(i'')_{M_U}\otimes(k'')_{K_U}} \circ\pi_A,\, \mathrm{G}_{(k'')_{K_U}\otimes(j'')_{N_U}} \circ\pi_B\Big). \end{aligned}$$

Base case. Now we further break down using rule (3.1) expanding

$$M''_{i,j,k,i',j',k',i'',j'',k''} \to \pi_C + \mathrm{S}_{f_{i,j,k,i',j',k',i'',j'',k''}} \circ \big(\pi_A \cdot \pi_B\big)$$

with

$$f_{i,j,k,i',j',k',i'',j'',k''} = (i)_{N/N_B} \otimes (i')_{N_B/M_U} \otimes (i'')_{M_U} \otimes (j)_{M/N_B} \otimes (j')_{N_B/N_U} \otimes (j'')_{N_U}$$

Backsubstitution 1. Substituting the register blocking into the L1 blocking produces the expression

$$\underbrace{\pi_C \circ \left(\prod_{j=0}^{\frac{N}{N_B}-1} \prod_{i=0}^{\frac{M}{N_B}-1} \prod_{k=0}^{\frac{K}{N_B}-1} \left(\pi_C \circ \left(\prod_{j'=0}^{\frac{N_B}{N_U}-1} \prod_{i'=0}^{\frac{N_B}{M_U}-1} \prod_{k'=0}^{\frac{N_B}{K_U}-1} \left(M'_{i,j,k,i',j',k'} \circ G'_{i',j',k'},\, \pi_A,\, \pi_B \right) \right) \circ G_{i,j,k},\, \pi_A,\, \pi_B \right) \right)}_{\text{inplace } (C,A,B)\to C}$$

Applying rewriting rule (1.3) leads to

$$\underbrace{\pi_C \circ \left(\prod_{j=0}^{\frac{N}{N_B}-1} \prod_{i=0}^{\frac{M}{N_B}-1} \prod_{k=0}^{\frac{K}{N_B}-1} \left(\pi_C \circ \left(\prod_{j'=0}^{\frac{N_B}{N_U}-1} \prod_{i'=0}^{\frac{N_B}{M_U}-1} \prod_{k'=0}^{\frac{N_B}{K_U}-1} \left(M'_{i,j,k,i',j',k'} \circ \hat{G}_{i,j,k,i',j',k'},\, \pi_A,\, \pi_B \right) \right),\, \pi_A,\, \pi_B \right) \right)}_{\text{inplace } (C,A,B)\to C}$$

with

$$\hat{G}_{i,j,k,i',j',k'} = \Big(\pi_C,\ \mathrm{G}_{(i')_{N_B/M_U}\otimes\imath_{M_U}\otimes(k')_{N_B/K_U}\otimes\imath_{K_U}}\circ\pi_A,\ \mathrm{G}_{(k')_{N_B/K_U}\otimes\imath_{K_U}\otimes(j')_{N_B/N_U}\otimes\imath_{N_U}}\circ\pi_B\Big)\circ$$
$$\Big(\pi_C,\ \mathrm{G}_{(i)_{N/N_B}\otimes\imath_{N_B}\otimes(k)_{K/N_B}\otimes\imath_{N_B}}\circ\pi_A,\ \mathrm{G}_{(k)_{K/N_B}\otimes\imath_{N_B}\otimes(j)_{M/N_B}\otimes\imath_{N_B}}\circ\pi_B\Big).$$

Applying rule (1.4) leads to the expression

$$\underbrace{\pi_C\circ\left(\prod_{j=0}^{\frac{N}{N_B}-1}\prod_{i=0}^{\frac{M}{N_B}-1}\prod_{k=0}^{\frac{K}{N_B}-1}\prod_{j'=0}^{\frac{N_B}{N_U}-1}\prod_{i'=0}^{\frac{N_B}{M_U}-1}\prod_{k'=0}^{\frac{N_B}{K_U}-1}\Big(M'_{i,j,k,i',j',k'}\circ\hat{G}_{i,j,k,i',j',k'},\ \pi_A,\ \pi_B\Big)\right)}_{\text{inplace }(C,A,B)\to C}$$

Applying rules (1.1) and (1.2) simplify $\hat{G}_{i,j,k,i',j',k'}$ further:

$$\begin{aligned}\hat{G}_{i,j,k,i',j',k'} = \Big(&\pi_C,\\ &\mathrm{G}_{(i)_{N/N_B}\otimes(i')_{N_B/M_U}\otimes\imath_{M_U}\otimes(k)_{K/N_B}\otimes(k')_{N_B/K_U}\otimes\imath_{K_U}}\circ\pi_A,\\ &\mathrm{G}_{(k)_{K/N_B}\otimes(k')_{N_B/K_U}\otimes\imath_{K_U}\otimes(j)_{M/N_B}\otimes(j')_{N_B/N_U}\otimes\imath_{N_U}}\circ\pi_B\Big).\end{aligned}$$

Backsubstitution 2. Substituting unrolling and applying the same rewriting rules leads to

$$\underbrace{\pi_C\circ\left(\prod_{j=0}^{\frac{N}{N_B}-1}\prod_{i=0}^{\frac{M}{N_B}-1}\prod_{k=0}^{\frac{K}{N_B}-1}\prod_{j'=0}^{\frac{N_B}{N_U}-1}\prod_{i'=0}^{\frac{N_B}{M_U}-1}\prod_{k'=0}^{\frac{N_B}{K_U}-1}\prod_{k''=0}^{K_U-1}\prod_{j''=0}^{N_U-1}\prod_{i''=0}^{M_U-1}\Big(M''_{i,j,k,i',j',k',i'',j'',k''}\circ\tilde{G}_{i,j,k,i',j',k',i'',j'',k''},\ \pi_A,\ \pi_B\Big)\right)}_{\text{inplace }(C,A,B)\to C}$$

with

$$\begin{aligned}\tilde{G}_{i,j,k,i',j',k',i'',j'',k''} &= \big(\pi_C,\ \mathrm{G}_g\circ\pi_A,\ \mathrm{G}_h\circ\pi_B\big)\\ g &= (i)_{N/N_B}\otimes(i')_{N_B/M_U}\otimes(i'')_{M_U}\otimes(k)_{K/N_B}\otimes(k')_{N_B/K_U}\otimes(k'')_{K_U}\\ h &= (k)_{K/N_B}\otimes(k')_{N_B/K_U}\otimes(k'')_{K_U}\otimes(j)_{M/N_B}\otimes(j')_{N_B/N_U}\otimes(j'')_{N_U}\,.\end{aligned}$$

Final expression. Substituting the base case and applying the same rules as in the naive example leads to the final expression,

$$\underbrace{\pi_C\circ\left(\prod_{j=0}^{\frac{N}{N_B}-1}\prod_{i=0}^{\frac{M}{N_B}-1}\prod_{k=0}^{\frac{K}{N_B}-1}\prod_{j'=0}^{\frac{N_B}{N_U}-1}\prod_{i'=0}^{\frac{N_B}{M_U}-1}\prod_{k'=0}^{\frac{N_B}{K_U}-1}\prod_{k''=0}^{K_U-1}\prod_{j''=0}^{N_U-1}\prod_{i''=0}^{M_U-1}\Big(\pi_C+\mathrm{S}_f\circ\big((\mathrm{G}_g\circ\pi_A)\cdot(\mathrm{G}_h\circ\pi_B)\big),\ \pi_A,\ \pi_B\Big)\right)}_{\text{inplace }(C,A,B)\to C}$$

with

$$\begin{aligned}f &= (i)_{N/N_B}\otimes(i')_{N_B/M_U}\otimes(i'')_{M_U}\otimes(j)_{M/N_B}\otimes(j')_{N_B/N_U}\otimes(j'')_{N_U}\\ g &= (i)_{N/N_B}\otimes(i')_{N_B/M_U}\otimes(i'')_{M_U}\otimes(k)_{K/N_B}\otimes(k')_{N_B/K_U}\otimes(k'')_{K_U}\\ h &= (k)_{K/N_B}\otimes(k')_{N_B/K_U}\otimes(k'')_{K_U}\otimes(j)_{M/N_B}\otimes(j')_{N_B/N_U}\otimes(j'')_{N_U}\,.\end{aligned}$$

This is the Σ-SPL equivalent to ATLAS as explained in [2]:

$$
\begin{array}{l}
\texttt{for}\ j \in [1 : N_B : M] \\
\quad \texttt{for}\ i \in [1 : N_B : N] \\
\quad\quad \texttt{for}\ k \in [1 : N_B : K] \\
\quad\quad\quad \texttt{for}\ j' \in [j : N_U : j + N_B - 1] \\
\quad\quad\quad \texttt{for}\ i' \in [i : M_U : i + N_B - 1] \\
\quad\quad\quad \texttt{for}\ k' \in [j : K_U : k + N_B - 1] \\
\quad\quad\quad\quad \texttt{for}\ k'' \in [k' : 1 : k' + K_U - 1] \\
\quad\quad\quad\quad \texttt{for}\ j'' \in [j' : 1 : j' + N_U - 1] \\
\quad\quad\quad\quad \texttt{for}\ i'' \in [i' : 1 : i' + M_U - 1] \\
\quad\quad\quad\quad\quad C_{i'',j''} = C_{i'',j''} + A_{i''k''} B_{k''j''}
\end{array}
$$

References

[1] F. Franchetti, Y. Voronenko, and M. Püschel. Loop merging for signal transforms. In *Proc. Programming Language Design and Implementation (PLDI)*, pages 315–326, 2005.

[2] Kamen Yotov, Xiaoming Li, Gang Ren, Maria Garzaran, David Padua, Keshav Pingali, and Paul Stodghill. A comparison of empirical and model-driven optimization. *Proceedings of the IEEE*, 93(2), 2005. Special issue on "Program Generation, Optimization, and Adaptation".