Vectorization of Sorting Networks

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1 Bitonic Search

2 OL for Sorting Networks

2.1 Definitions

We define two sorting operators:

Definition 1 (Ascending Sort Operator)

$$\chi_n : \begin{cases} \mathbb{R}^n \to \mathbb{R}^n \\ (a_i)_{0 \le i < n} \mapsto (a_{\sigma(i)})_{0 \le i < n}, \ a_{\sigma(j)} \le a_{\sigma(k)} \ \textit{for } j \le k \end{cases}$$

Definition 2 (Descending Sort Operator)

$$\Theta_n : \begin{cases} \mathbb{R}^n \to \mathbb{R}^n \\ (a_i)_{0 \le i < n} \mapsto (a_{\sigma(i)})_{0 \le i < n}, \ a_{\sigma(j)} \ge a_{\sigma(k)} \ \textit{for } j \le k \end{cases}$$

2.2 New Identities

For $n = 2$ sorting operators can be expressed using $\min(.,.)$ and $\max(.,.)$.

Corollary 1 (Min-Max Comparator)

$$\Theta_2 : \begin{cases} \mathbb{R}^2 \to \mathbb{R}^2 \\ \big(a, b\big) \mapsto \big(\min(a, b), \ \max(a, b)\big) \end{cases}$$

Corollary 2 (Max-Min Comparator)

$$\chi_2 : \begin{cases} \mathbb{R}^2 \to \mathbb{R}^2 \\ \big(a, b\big) \mapsto \big(\max(a, b), \ \min(a, b)\big) \end{cases}$$

Definition 3 (Standard OL Operators)

$$\mathrm{I}_n \tag{1}$$

$$\mathrm{J}_n \tag{2}$$

$$\mathrm{L}^{mn}_m \tag{3}$$

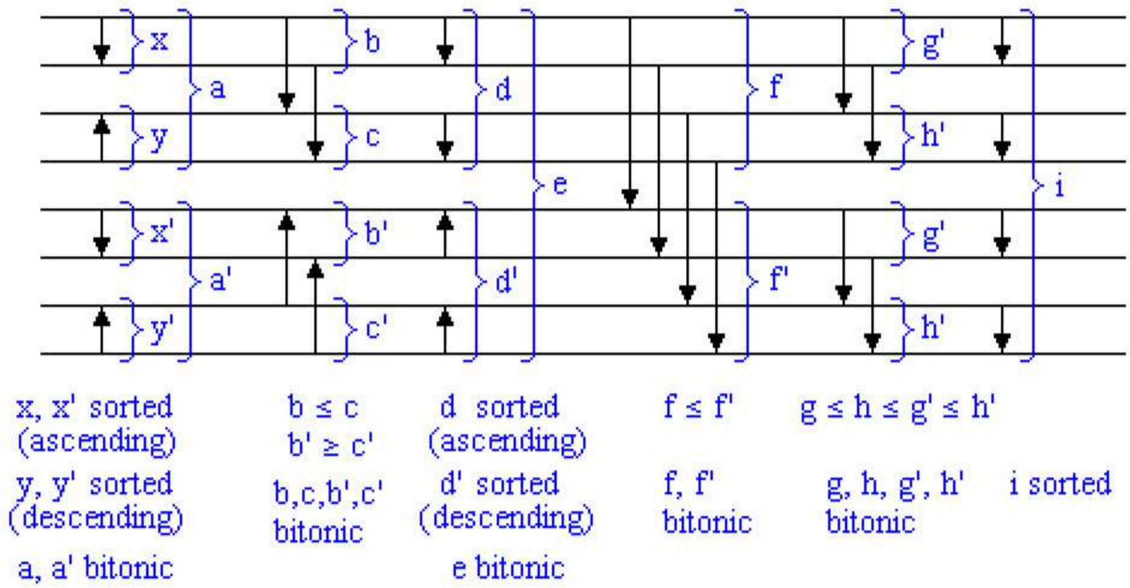

Figure 1: Bitonic Sorting Network for 8 elements.

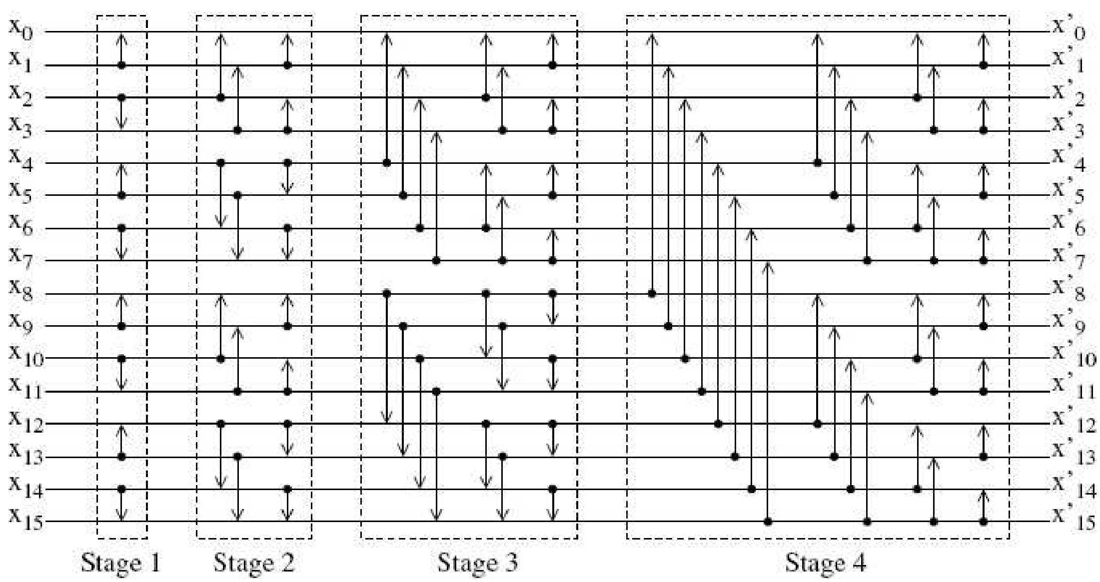

Figure 2: Bitonic Sorting Network for 16 elements.

Definition 4 (Standard OL Operations)

$$A \oplus B \tag{4}$$
$$\mathrm{I}_n \otimes A \tag{5}$$
$$A \otimes \mathrm{I}_n \tag{6}$$
$$A \circ B \tag{7}$$

$$\chi_n = \mathrm{J}_n \circ \Theta_n \tag{8}$$
$$\Theta_n = \mathrm{J}_n \circ \chi_n \tag{9}$$
$$\chi_n = (-\mathrm{I}_n) \circ \Theta_n \circ (-\mathrm{I}_n) \tag{10}$$
$$\Theta_n = (-\mathrm{I}_n) \circ \chi_n \circ (-\mathrm{I}_n) \tag{11}$$

3 Bitonic Sorting in OL

3.1 Example: Bitonic Sorting of 16 Elements in OL

$$\begin{aligned} \chi_{16} \rightarrow\ & \Big(\mathrm{I}_8 \otimes \Theta_2\Big) \circ \Big(\mathrm{I}_4 \otimes \big(\Theta_2 \otimes \mathrm{I}_2\big)\Big) \circ \Big(\mathrm{I}_2 \otimes \big(\Theta_2 \otimes \mathrm{I}_4\big)\Big) \circ \Big(\Theta_2 \otimes \mathrm{I}_8\Big) \\ & \circ \Big(\big(\mathrm{I}_4 \otimes \Theta_2\big) \oplus \big(\mathrm{I}_4 \otimes \chi_2\big)\Big) \circ \Big(\big(\mathrm{I}_2 \otimes (\Theta_2 \otimes \mathrm{I}_2)\big) \oplus \big(\mathrm{I}_2 \otimes (\chi_2 \otimes \mathrm{I}_2)\big)\Big) \circ \Big((\Theta_2 \otimes \mathrm{I}_4) \oplus (\chi_2 \otimes \mathrm{I}_4)\Big) \\ & \circ \Big(\mathrm{I}_2 \otimes \big((\mathrm{I}_2 \otimes \Theta_2) \oplus (\mathrm{I}_2 \otimes \chi_2)\big)\Big) \circ \Big(\mathrm{I}_2 \otimes \big((\Theta_2 \otimes \mathrm{I}_2) \oplus (\chi_2 \otimes \mathrm{I}_2)\big)\Big) \\ & \circ \Big(\mathrm{I}_4 \otimes \big(\Theta_2 \oplus \chi_2\big)\Big) \end{aligned}$$

3.2 Bitonic Sorting as OL Breakdown Rules

Definition 5 (Bitonic Merge) *For (a_i) and (c_i) sorted ascending and (b_i) sorted descending, bitonic merge is defined as follows:*

$$M_n : \begin{cases} \mathbb{R}^n \rightarrow \mathbb{R}^n \\ (a_i)_{0 \le i < n/2} \oplus (b_i)_{0 \le i < n/2} \mapsto (c_i)_{0 \le i < n} \end{cases}$$

Rules 1 (Recursive Bitonic Sort)

$$\begin{aligned} \chi_n &\rightarrow M_n \circ \big(\chi_{n/2} \oplus \Theta_{n/2}\big) \\ M_n &\rightarrow \big(\mathrm{I}_2 \otimes M_{n/2}\big) \circ \big(\Theta_2 \otimes \mathrm{I}_{n/2}\big) \end{aligned}$$

3.3 Example: 2-way Vectorization of χ_4

$$\begin{aligned} \chi_4 &= \big(\mathrm{I}_2 \otimes \Theta_2\big) \circ \big(\Theta_2 \otimes \mathrm{I}_2\big) \circ \big(\Theta_2 \oplus \chi_2\big) \\ &= \big(\mathrm{I}_2 \otimes \Theta_2\big) \circ \big(\Theta_2 \otimes \mathrm{I}_2\big) \circ (\mathrm{I}_2 \oplus \mathrm{J}_2) \circ (\mathrm{I}_2 \otimes \Theta_2) \\ &= \mathrm{L}^4_2 \circ \big(\Theta_2 \otimes \mathrm{I}_2\big) \circ \mathrm{L}^4_2 \circ \big(\Theta_2 \otimes \mathrm{I}_2\big) \circ \big((\mathrm{I}_2 \oplus \mathrm{J}_2) \circ \mathrm{L}^4_2\big) \circ \big(\Theta_2 \otimes \mathrm{I}_2\big) \circ \mathrm{L}^4_2 \end{aligned}$$